BOOK REVIEWS

Foundations of algebraic geometry. By André Weil. (American Mathematical Society Colloquium Publications, vol. 29.) New York, American Mathematical Society, 1946. 20+288 pp. $5.50.

In the words of the author the main purpose of this book is "to present a detailed and connected treatment of the properties of intersection multiplicities, which is to include all that is necessary and sufficient to legitimize the use made of these multiplicities in classical algebraic geometry, especially of the Italian school." There can be no doubt whatsoever that this purpose has been fully achieved by Weil. After a long and careful preparation (Chaps. I–IV) he develops in two central chapters (V and VI) an intersection theory which for completeness and generality leaves little to be desired. It goes far beyond the previous treatments of this foundational topic by Severi and van der Waerden and is presented with that absolute rigor to which we are becoming accustomed in algebraic geometry. In harmony with its title the book is entirely self-contained and the subject matter is developed *ab initio.*

It is a remarkable feature of the book that—with one exception (Chap. III)—no use is made of the higher methods of modern algebra. The author has made up his mind not to assume or use modern algebra "beyond the simplest facts about abstract fields and their extensions and the bare rudiments of the theory of ideals." Weil's faithful realization of this program of strict mathematical economy is an achievement in itself. In some cases this leads to the "best possible" proofs. However, on the whole one may question the wisdom of this self-imposed régime of austerity. The methodical reduction of the theory—which is both difficult and subtle—to the primitives of algebra is bound to be a very laborious process. As a result, the reader finds himself very much in the position of a man who must collect a large amount of cash most of which is in pennies. The author justifies his procedure by an argument of historical continuity, urging a return "to the palaces which are ours by birthright." But it is very unlikely that our predecessors will recognize in Weil's book their own familiar edifice, however improved and completed. If the traditional geometer were invited to choose between "makeshift constructions full of rings, ideals and valuations" on one hand, and constructions full of fields, linearly disjoint fields, regular extensions, independent extensions, generic specializations, finite specializations and specializations of specializations on the other, he most probably would decline the choice and say: "A plague on both your houses!" That being so, we may just as well help ourselves to modern algebra to the fullest possible extent.

To achieve his objectives Weil wages a campaign of the Satz-Beweis type. Most readers will find it difficult to follow the author through the seemingly endless series of propositions, theorems, lemmas and corollaries (their total must be close to 300). A further obstacle to continuous reading of the book are the numerous cross-references in the proofs, such as "by coroll. 2 of th. 2, §1" or "by th. 5, Ch. I, §7" (for instance, the 13 lines of proof of proposition 7, p. 33, contain 8 such references). However, this being said, the fact remains that the scientific value of this surprisingly large collection of theorems and propositions is very high. The specialist, or the advanced student who is interested in some particular phase of the theory, will frequently be able to find in Weil's book just the theorem or lemma he needs.

The first chapter contains algebraic preliminaries. The concept of linearly disjoint fields is introduced and used with good effect for a study of inseparable field extensions. Of particular importance later on are the "regular extensions" of a field k: if (x) is a finite set of quantities, the field $k(x)$ is a regular extension of k if it is separably generated over k and if k is maximally algebraic in $k(x)$.

The "algebraic theory of specializations" is the topic of the second chapter, and it plays a fundamental role in Weil's intersection theory. Here the main result is Theorem 6 which asserts that if (x) and (y) are two (finite) sets of quantities, then any specialization $(x)\rightarrow(x')$ of the set (x) (with reference to a given field k) can be extended to a specialization $(x, y)\rightarrow(x', y')$ of the combined set (x, y). This well known result from the theory of algebraic correspondences is proved here without the use of elimination theory. It must be observed that Weil deals only with varieties in the affine space and never uses homogeneous coördinates. Therefore the symbol ∞ is allowed to occur among his quantities when he discusses specializations. As a matter of fact, projective models are not varieties at all in Weil's sense; they first occur as an afterthought only at the very end of the book (Appendix I).

The principal aim of the next chapter is the concept of the multiplicity of a specialization. Assume that the quantities x are independent variables over k and that the y's are algebraic over $k(x)$. Let V be the algebraic variety having (x, y) as general point over k and let $(\bar{x}, \bar{y})$ be a point of V [whence $(x, y)\rightarrow(\bar{x}, \bar{y})$ is a specialization over k] such that the $\bar{y}$ are algebraic over $k(\bar{x})$. Consider a complete set of conjugates $(y^{(1)}, y^{(2)}, \cdots, y^{(n)})$ of (y) over $k(x)$ and extend in any manner whatsoever the specialization $(x)\rightarrow(\bar{x})$ to a specialization $(x, y^{(1)}, y^{(2)}, \cdots, y^{(n)})\rightarrow(\bar{x}, \bar{y}^{(1)}, \bar{y}^{(2)}, \cdots, \bar{y}^{(n)})$ over k. The main result is as follows: if the intersection $(\bar{x}, \bar{y})$ of the linear space $X=\bar{x}$ with V is *isolated*, then $\bar{y}$ occurs the same number of times in every set $(\bar{y}^{(1)}, \bar{y}^{(2)}, \cdots, \bar{y}^{(n)})$ such as above, and this number is positive (Proposition 7 and Theorem 4). Weil calls this number the multiplicity of $(\bar{y})$ as a specialization of (y) over $(x)\rightarrow(\bar{x})$, with reference to k. Essentially this number is the intersection multiplicity of V with the linear space L: $X=\bar{x}$, at the point $P(\bar{x}, \bar{y})$. This clears the ground completely for the "special case" of the intersection theory, dealt with in Chap. V, where intersection multiplicities of varieties with linear spaces are studied in greater detail. It should be observed that Weil does not assume that *all* the intersections of V and L are of the "right" dimension, that is, of dimension 0 over $k(\bar{x})$, and therefore the result is purely of local character. Therein lies the main difficulty of the proof. But at the same time it is precisely because of its local validity that the theory developed by Weil (as well as the one developed independently by Chevalley, Trans. Amer. Math. Soc. (1945)) is an improvement over the previous treatments of intersection multiplicities. In this chapter the author is forced to use "higher" methods of modern algebra (ideal theory of Noetherian domains, rings of formal power series, and so on). Later on in Chap. IX (*Comments and discussions*) he shows that the main result of the present chapter can also be derived from a theorem on birational correspondences due to the reviewer. Our proof involves a good deal of ideal theory of integrally closed domains. All this illustrates the plain truth that as one advances in the field of algebraic geometry, the plot "thickens" and more powerful tools of modern algebra become indispensable.

In Chap. IV algebraic varieties are introduced, linear varieties and simple points are discussed, and the properties of product varieties are developed. The author restricts his class of varieties V by the following condition: there must exist a field k and a point P of V such that P is a general point of V with respect to *k and the field $k(P)$ is a regular extension of k.* Furthermore he restricts the concept of a simple point by defining it in terms of the classical Jacobian criterion. This enables him to state and prove his theorems without any reference to any particular ground field, for it is well known that with the above restrictions nothing much can happen to a variety or to a simple point when the ground field undergoes an arbitrary extension. These limitations are perfectly legitimate and undoubtedly create a fortunate state of affairs, but we could not comprehend Weil's contention that they entail an exclusive point of view that emphasizes "the geometric content of all notions." At any rate it would be desirable to free the intersection theory from all these restrictions. This theory is developed in the next two chapters, V and VI. The main object is of course the introduction and study of the main properties of the intersection multiplicity of two subvarieties A and B of a given variety U, along an irreducible component C of $A\cap B$. This multiplicity, denoted by $i(A\cdot B,\ C;\ U)$, is to be defined only under the following conditions: C is simple for U and has the right dimension ($=\dim A+\dim B-\dim U$). We have already mentioned the "special case": U and one of the varieties A and B are linear. The general case is reduced to the special case by using product varieties. Thus, if U is still linear and C is a point P, say the origin, then the above symbol is defined as $i(A\times B\cdot\Delta,\ P\times P;\ U\times U)$, where Δ is the diagonal of $U\times U$. In the most general case the reduction is quite similar. There follows an exhaustive study of the properties of intersection multiplicities. The basic properties are stated in Theorems 5, 6 and 8, Ch. VI, which express respectively the associative law, the projection formula and the criterion of multiplicity 1.

The long Chap. VII is dedicated entirely to what Weil calls "abstract varieties," or Varieties with a capital V. A Variety is a finite collection $\{V_\alpha-F_\alpha\}$ of pieces of birationally equivalent varieties V_α such that: (1) F_α is a subvariety of V_α: (2) whenever two pieces $V_\alpha-F_\alpha$, $V_\beta-F_\beta$ contain corresponding points of V_α and V_β, the local rings of these points coincide. A Variety is *complete* if it satisfies a suitable condition of closure. Thus projective models are complete Varieties. The chapter deals primarily with the extension of the results of the preceding chapters to Varieties.

We find it very difficult to estimate the necessity or the permanence value of this new concept. The book contains no examples of complete Varieties other than projective models. It is true that the realization of an Abelian variety by means of a *nonsingular* complete Variety follows immediately from the definition of Abelian varieties, while the existence of a corresponding nonsingular projective model still has to be proved (in the classical case this has been proved by Lefschetz). But we point out that there exists no proof that every function field can be realized by a nonsingular complete Variety. Moreover, we believe that, as to degree of difficulty, such a proof would differ only by an ϵ from a proof of the stronger statement that every function field possesses a nonsingular projective model. Indeed, it may be conjectured that every complete Variety is itself a projective model. In view of all this, one wonders whether Chap. VII is absolutely necessary for the *general* theory of algebraic varieties, or even for the theory of Abelian varieties.

Weil's book is the first purely arithmetic exposition of an important sector of algebraic geometry, and is therefore a landmark in the literature of this field. This, and the competence of the author, give the book added significance.

In the remainder of the book we wish to recommend especially the interesting chapter entitled *Comments and discussions*, where various unsolved problems are discussed and possible directions of future research are indicated. The book has an excellent list of definitions and table of notations.

OSCAR ZARISKI

Transformations on lattices and structures of logic. By S. A. Kiss. New York, 1947. 10+322 pp. $5.00.

This volume by Dr. Kiss is an instance of a rare phenomenon—a significant contribution to mathematics by one who is not a professional mathematician. Dr. Kiss is a patent lawyer by profession, with an advanced degree in chemistry, who has here enriched the algebra of logic by a new idea, and developed the idea in full detail. Moreover enough elementary material has been adapted from standard sources so that the book is self-contained as regards both algebra and logic. The book was published by the author, and can be obtained from him at 11 E. 92nd Street, New York City.

Boolean algebra may be described as the algebra of true and false. Numerous equivalent postulate systems for it are known, involving from one to three undefined operations or relations, in terms of which all 2^{2^n} possible n-ary operations on a two-element system can be defined. Thus *every finitary operation on a two-element system is a Boolean operation.*

But the corresponding result for the four-element Boolean algebra B^2 does not hold. Only 2^{2^2} of the 4^{4^4} possible binary operations on B^2 are "Boolean," in the usual sense. Dr. Kiss proposes the following ingenious extension. Consider the self-dual ternary operation $(x, y, z) = (x\cup y)\cap(y\cup z)\cap(z\cup x)$. In terms of this, joins and meets can be defined by fixing y as one of the two logical constants 0 and I: $x\cup z = (x, I, z)$ and $x\cap z = (x, 0, z)$. If we use the other two constants e and e' of B^2 in place of 0 and I, we get two further binary Boolean-like operations (x, e, z) and (x, e', z). And in terms of these, all binary operations can be defined.

It is still too early to appraise the ultimate importance of these and