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ON CONFIRMATION

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1. Taking for the relation of confirmation the following obvious axioms, we obtain several more or less well-known theorems and are able to solve in a definite and strict manner several problems concerning confirmation.

Let a, b, and c be variable names of sentences belonging to a certain class,[1] the operations $a \cdot b$, $a + b$, and $\bar{a}$ the (syntactical) product, the sum, and the negation of them. Let us further assume the existence of a real non-negative function $\mathfrak{c}(a, b)$ of a and b, when b is not self-contradictory. Let us read '$\mathfrak{c}(a, b)$' 'degree of confirmation of a with respect to b' and take the following axioms:

Axiom I. If a is a consequence of b, $\mathfrak{c}(a, b) = 1$.

Axiom II. If $\overline{a \cdot b}$ is a consequence of c,

$$\mathfrak{c}(a+b, c) = \mathfrak{c}(a, c) + \mathfrak{c}(b, c).$$

Axiom III. $\mathfrak{c}(a \cdot b, c) = \mathfrak{c}(a, c) \cdot \mathfrak{c}(b, a \cdot c)$.

Axiom IV. If b is equivalent to c, $\mathfrak{c}(a, b) = \mathfrak{c}(a, c)$.[2]

As may be easily seen, the interval of variation for $\mathfrak{c}$ is $(0, 1)$; this is quite conventional.

2. We begin with some simple propositions about confirmation, $f_1)$–$f_6)$ below, which follow from the above axioms. These propositions can be derived with the help of a proposition which is easily obtained from the axioms, i.e.:

$$\text{(I)}\quad \mathfrak{c}(b, a \cdot c) = \frac{\mathfrak{c}(b, c)}{\mathfrak{c}(a, c)} \text{ if } a \text{ is a consequence of } b \text{ and } c \text{ and if } \mathfrak{c}(a, c) \neq 0.$$

Let c represent our knowledge at a given moment, b a sentence (hypothesis, law, or the like) in whose $\mathfrak{c}$ we are interested, and a the statement of an observed fact (observed just after the given moment at which c represents our knowledge) which is a consequence of b and c.

I shall frequently omit the characterization "consequence of b and c" and simply speak about facts a and their law or hypothesis b, understanding by this statements of facts a which are consequences of the law or hypothesis b and our present knowledge c. Similarly, I shall speak about a hypothesis or law and its facts.

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[1] The class must be broad enough to include all sentences for which we desire to speak about confirmation.

[2] These axioms are analogous to St. Mazurkiewicz's system of axioms for probabilities (see *Zur Axiomatik der Wahrscheinlichkeitsrechnung*, ***Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie***, vol. 25 (1932)).

f_1) A sentence b which is certain "a priori" i.e. certain on the basis of our knowledge at the given moment, cannot increase or decrease its $\mathfrak{c}$ under the influence of its observed fact a, unless $\mathfrak{c}(a, c) = 0$.

For, if $\mathfrak{c}(b, c) = 1$, then $\mathfrak{c}(b, a \cdot c) = \dfrac{1}{\mathfrak{c}(a, c)}$ by (I); but neither $\mathfrak{c}(b, a \cdot c)$ nor $\mathfrak{c}(a, c)$ can exceed 1, and therefore the theorem follows. Thus the $\mathfrak{c}$ of the sentence 'salt is salt,' if this sentence is certain a priori, cannot increase or decrease under the observation of a bit of salt which is salt.

f_2) A sentence b, false a priori, cannot increase or decrease its $\mathfrak{c}$ under the influence of its observed fact, unless $\mathfrak{c}(a, c) = 0$.

For 0 divided by any number different from 0 is 0. E.g., the $\mathfrak{c}$ of the sentence 'All children born this year are girls' cannot increase on the basis of the fact that a given child born this year is a girl.

f_3) A fact which is certain a priori does not increase or decrease the $\mathfrak{c}$ of its law or hypothesis.

For, if $\mathfrak{c}(a, c) = 1$, $\mathfrak{c}(b, a \cdot c) = \mathfrak{c}(b, c)$. E.g., if the fact that this salt is salt is certain a priori, it does not raise or decrease the $\mathfrak{c}$ of its law.

f_4) The smaller is the a priori $\mathfrak{c}$ of a fact, the more does the $\mathfrak{c}$ of its law or hypothesis increase when this fact is observed.[3]

For the smaller is $\mathfrak{c}(a, c)$, the greater is $\mathfrak{c}(b, a \cdot c)$. E.g., if a weather forecast for a whole week, deduced from meteorological data, turns out to be true, the meteorological assumptions are confirmed to a greater degree than a similar fulfilled weather forecast concerning only one day of this week.

f_5) The same fact, on the basis of the same body of knowledge, raises by a smaller value the $\mathfrak{c}$ of its law if this $\mathfrak{c}$ is smaller a priori.

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[3] This fact may be expressed more simply and intuitively as follows. We find in chapters on induction and probability the statement that *a hypothesis is the more probable the more facts we have observed following from it.* This statement is a special case of f_4) if we take the degree of confirmation instead of probability. For, the product of facts $a_1 \cdot a_2 \cdots a_n$ has a smaller $\mathfrak{c}$ than the product $a_1 \cdot a_2 \cdots a_{n-1}$—a result easily obtained on the basis of our axioms, provided that a_n does not follow from $a_1 \cdot a_2 \cdots a_{n-1}$. (If a_n does follow, we would not say that we have observed *more* facts, when a_n is observed after $a_1 \cdot a_2 \cdots a_{n-1}$.)

Moreover, assuming that we obtain more or stronger knowledge or data by observing or stating facts which were more difficult to anticipate, i.e. less confirmed a priori, we may simply define "more or stronger observed facts or data" by "less confirmed a priori." Let us therefore say that: F_1 are more or stronger observed facts or data than $F_2 \equiv \mathfrak{c}(F_1, c) < \mathfrak{c}(F_2, c)$, where c is the knowledge available at the time. This definition allows us to compare—with respect to more or stronger facts or data—not only two sets of facts where the first includes or implies the other, but also two quite independent sets of facts. E.g. an observation of the weather during a whole week constitutes, *ceteris paribus*, more or stronger facts or data according to our definition than an observation of the weather during one day of *another* week.

According to this definition, however, f_4) becomes equivalent to the simpler and more intuitive statement given at the beginning of this note (in italics). We have only to substitute in it "confirmed" instead of "probable."

Suppose $\mathfrak{c}(a, c) = p$, with $0 < p < 1$, and $\mathfrak{c}(b, c) = s$. Then, since by (I) $\mathfrak{c}(b, a \cdot c) = \frac{s}{p}$, the difference between the a posteriori and a priori values of the $\mathfrak{c}$ of the law is $\frac{s}{p} - s$. But $\frac{s}{p} - s = s\left(\frac{1}{p} - 1\right)$, so that as s increases so does this difference. Thus the $\mathfrak{c}$ of a hypothesis with a small a priori $\mathfrak{c}$ approaches 1 (certainty) more slowly when based upon an observed instance of it, than does the $\mathfrak{c}$ of a hypothesis with a greater a priori $\mathfrak{c}$ on the basis of the same instance. The approach of degrees of confirmation to 1 therefore proceeds like an avalanche.[4] E.g., we take a toadstool, inject into it a certain substance S, whose influence we do not know. We can only see that the color of this substance is similar to a known poison. Subsequently we discover that this toadstool was eaten by somebody who becomes poisoned. Then, the $\mathfrak{c}$ of the law 'Toadstools are poisonous' differs less from certainty than the $\mathfrak{c}$ of the hypothesis 'Substances S are poisonous.'

$\mathfrak{f}_6$) The following criticism of procedures of confirmation has been made, though it is sometimes possible to ignore it: The $\mathfrak{c}$ of a sentence b should always be related to (based upon) data c_1 which are directly confirmed, rather than to observed physical facts a which are in their turn based upon c_1; for only c_1 can be counted as belonging to our established knowledge, which is fixed once for all.

For example, let b be the sentence: 'Water boils at the temperature of 98.5°C, under the pressure of 720 mm.' Let the fact a be observed: On a given thermometer placed into boiling water under the pressure of 720 mm, the mercury reaches 98.5°C. Then, according to the above criticism, the directly confirmed datum is either c_1: I see such and such shapes coinciding; or (purely phenomenologically) c_1: Such and such coinciding shapes are given. Consequently, we should base the $\mathfrak{c}$ of b upon c_1 instead of upon a (and c_1).

Now, if $\mathfrak{c}(a, c_1 \cdot c)$ is high enough (as in our example), $\mathfrak{c}(b, a \cdot c_1 \cdot c)$ is close to $\mathfrak{c}(b, c_1 \cdot c)$; and the greater is $\mathfrak{c}(a, c_1 \cdot c)$, the nearer is $\mathfrak{c}(b, a \cdot c_1 \cdot c)$ to $\mathfrak{c}(b, c_1 \cdot c)$. To see this it is enough to substitute $c_1 \cdot c$ for c in (I) and remember that a follows from $b \cdot c$.

Therefore, when $\mathfrak{c}(a, c_1 \cdot c)$ is high enough, it is indifferent whether we consider $\mathfrak{c}(b, a \cdot c_1 \cdot c)$ or $\mathfrak{c}(b, c_1 \cdot c)$. However, it is safer to relate the $\mathfrak{c}$ of a sentence b to facts a whose $\mathfrak{c}$ with respect to $c_1 \cdot c$ is great, because the deviation from $\mathfrak{c}(b, c_1 \cdot c)$ is then small.

In spite of that we may relate b to facts a whose $\mathfrak{c}$ with respect to $c_1 \cdot c$ is not great, on the understanding that we realize the approximative character of our procedure and the extent of approximation. E.g., we may relate the $\mathfrak{c}$ of the sentence b_1, which gives a general law of the temperature of boiling water under a pressure, to the fact which constituted sentence b in the above example. We

[4] We may also say, however, that the distance from 1, when the $\mathfrak{c}$ of a hypothesis *decreases* under the influence of a fact, proceeds like an avalanche. If a fact is unfavorable for two hypotheses and its $\mathfrak{c}$ is equal with respect to both, then the difference in the distance from 1 of the $\mathfrak{c}$ will be greater for that one of the two hypotheses whose a priori $\mathfrak{c}$ is greater. This may be easily seen when applying (I′), see p. 144.

should remember, however, that the true $\mathfrak{c}$ of b_1 (true—in the opinion of objectors) would be only approximately the same as this.[5]

3. The propositions f_1)–f_6) may now be used to explain special problems, when certain further propositions concerning $\mathfrak{c}$ are assumed and some consequences of the above axioms are added. The following formula, a development of the denominator of the right-hand side of equation (I), is the most important of these consequences:

(II) If a follows from b and c, then $\mathfrak{c}(a, c) = \mathfrak{c}(b, c) + \mathfrak{c}(\bar{b}, c)\cdot\mathfrak{c}(a, \bar{b}\cdot c) = \mathfrak{c}(b, c) + \sum_{i=1}^{k} \mathfrak{c}(b_i, c)\cdot\mathfrak{c}(a, b_i\cdot c)$, where b_i are sentences (hypotheses) incompatible with one another and whose logical sum is $\bar{b}$.

Let us consider the following problems in which f_1)–f_6) will be used. We shall call the first one f_7), resolving it with the help of f_3) and f_4), and the second one f_8), resolving it with the help of f_5).

4. f_7) C. G. Hempel has stated the following paradox. The sentence a: 'This is a man, and is mortal,' confirms the general proposition b: 'Every man is mortal.' Moreover, the sentence a_1: 'This chair is not mortal, and is not a man,' confirms the general proposition b_1: 'No non-mortal being is a man.' Now b_1 is equivalent to b. Thus a_1, confirming b_1, should at the same time confirm b. But this sounds paradoxical.

The paradox disappears, I think, if we make use of proposition f_3) or f_4). Let us first apply f_3).

a_1 does not raise the $\mathfrak{c}$ of b_1 (nor, therefore, of b). For $\mathfrak{c}$ ('this is not a man', 'this is a chair and is not mortal'$\cdot c) = 1$, where c is the knowledge available to us at the given time, even without the addition of b. This is so because c contains the law that *no chair is a man.*[6]

But, it may be said in objection, for the confirmation of a sentence 'No $\bar{B}$ (non-B) is A' we are not forced to choose examples where it is known a priori that the given $\bar{B}$ is not A.

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[5] If a does not follow from $b\cdot c$, then

$$\frac{\mathfrak{c}(b,\ a\cdot c_1\cdot c)}{\mathfrak{c}(b,\ c_1\cdot c)} = \frac{\mathfrak{c}(a,\ b\cdot c_1\cdot c)}{\mathfrak{c}(a,\ c_1\cdot c)},$$

as we see from (I'), p. 144. Thus taking $\mathfrak{c}(b, a\cdot c_1\cdot c)$ instead of $\mathfrak{c}(b, c_1\cdot c)$ is the less dangerous, the nearer the right hand side of the equality is to *1*. If it is equal to *1*, we say that b is independent of a, given $c_1\cdot c$. In that case we can obviously take $\mathfrak{c}(b, a\cdot c_1\cdot c)$ instead of $\mathfrak{c}(b, c_1\cdot c)$, since they are equal.

[6] This emphasizes the fact that what may increase the $\mathfrak{c}$ of the instance of the law *Every A is B* is not the logical product *X is A and X is B*, but only its second factor (*X is B*) when the first (*X is A*) was observed, i.e. when *X is A* belongs to the knowledge we possess at the given time. For it is *X is B* which follows from *Every A is B and X is A* and not *X is A* from *Every A is B*. Thus, when I shall speak about the increase of the $\mathfrak{c}$ of a sentence on the basis of its instance, I shall mean the second factor of the instance, when the first was observed, i.e. when the first belongs to our knowledge already possessed.

It is rather difficult to find examples of a different type for our present b_1. But instead of continuing with b we may consider the sentence b': 'All salt (kitchen salt) is soluble in water.' And as a particular instance of the contrapositive of b' we take the statement a_1' that a certain substance S, which does not happen to be (kitchen) salt, is insoluble in water, although we do not know a priori that the substance is not (kitchen) salt. However, a_1' ('S is insoluble in water and is not salt') would still confirm the equivalent contrapositive of b', call it b_1': 'No substance insoluble in water is (kitchen) salt,' and so a_1' would also confirm b'. This still sounds paradoxical, although perhaps less so than before. For we would find it rather curious if a chemist, in order to confirm b', should take substances insoluble in water and then examine them to see if they are salt, instead of taking salts in order to discover whether they are soluble in water. Why would we find this strange? An application of $\mathfrak{f}_4$) to the case at hand supplies an answer, and we proceed to find it.

a_1' raises the $\mathfrak{c}$ of b_1', and therefore of b' also, quite negligibly in comparison with the increase of the $\mathfrak{c}$ of b' under the influence of a' (which asserts that a certain instance of salt is soluble in water). This is so for two reasons:

(i) The *number* of substances (or kinds of substances) insoluble in water is immense in comparison with the number of substances (or kinds of substances) which are salt.

It can be proved (see the demonstration below) on the basis of (II) that if the number of $\bar{B}$'s (non-B's) is greater than the number of A's, then the $\mathfrak{c}$ of an instance of $\bar{B}$ being $\bar{A}$ is, *ceteris paribus*, greater than the $\mathfrak{c}$ of an instance of A being B. Thus, on the strength of $\mathfrak{f}_4$), the second instance raises the $\mathfrak{c}$ of the sentence 'Every A is B' by a greater value than the first; and the difference between these two values increases with an increase in the ratio of the cardinal number of class $\bar{B}$ and of class A.

Demonstration: Let us read '$\text{NC}(X)$' as 'the number of X's,' and assume, in the first place, that a) $\text{NC}(A) = m$, $\text{NC}(\bar{B}) = n$, and $n>m$. We have to prove that the $\mathfrak{c}$ of a thing being B, when it is found to be A, is smaller than the $\mathfrak{c}$ of a thing being $\bar{A}$ when it is found to be $\bar{B}$. We have

$$(1) \qquad \text{NC}(\bar{B}\bar{A}) = \text{NC}(\bar{B}) - \text{NC}(A) + \text{NC}(AB)$$

(because $\text{NC}(\bar{B}\bar{A}) = \text{NC}(\bar{B}) - \text{NC}(\bar{B}A)$ and $\text{NC}(\bar{B}A) = \text{NC}(A) - \text{NC}(AB)$).

Let us call h_i the hypothesis that $\text{NC}(AB) = i$, for $i = 0, 1, 2, \cdots, m$; h_i is equivalent to $\text{NC}(\bar{B}\bar{A}) = n - m + i$ because of (1).

Let us now assume, in the second place, that b) $\mathfrak{c}(\text{'}X \text{ is } B\text{', '}X \text{ is } A\text{'}\cdot h_i\cdot c) = \dfrac{i}{m}$ and $\mathfrak{c}(\text{'}X \text{ is } \bar{A}\text{', '}X \text{ is } \bar{B}\text{'}\cdot h_i\cdot c) = \dfrac{n - m + i}{n}$, where c is our knowledge before establishing that X is A; i.e. all the A's have the same status with respect to being B, and all the $\bar{B}$'s have the same status with respect to being $\bar{A}$.

Now, on the basis of (II),

$$\mathfrak{c}(\text{'}X \text{ is } B\text{'}, \text{'}X \text{ is } A\text{'}\cdot c) = \sum_{i=0}^{m} \mathfrak{c}(h_i, \text{'}X \text{ is } A\text{'}\cdot c)\cdot\mathfrak{c}(\text{'}X \text{ is } B\text{'}, h_i\cdot\text{'}X \text{ is } A\text{'}\cdot c)$$
$$= \sum_{i=0}^{m} \mathfrak{c}(h_i, \text{'}X \text{ is } A\text{'}\cdot c)\cdot\frac{i}{m},$$
$$\mathfrak{c}(\text{'}X \text{ is } \bar{A}\text{'}, \text{'}X \text{ is } \bar{B}\text{'}\cdot c) = \sum_{i=0}^{m} \mathfrak{c}(h_i, \text{'}X \text{ is } \bar{B}\text{'}\cdot c)\cdot\frac{n-m+i}{n}.$$

Let us assume in the third place that c) $\mathfrak{c}(h_i, \text{'}X \text{ is } A\text{'}\cdot c) = \mathfrak{c}(h_i, \text{'}X \text{ is } \bar{B}\text{'}\cdot c) = \mathfrak{c}(h_i, c) = \mathfrak{c}_i$, for $i = 0, 1, 2, \cdots, m$. This assumption is the same as saying that the simple observation of an object being A (or $\bar{B}$) before we establish the fact that it is B (or $\bar{A}$) does not change the $\mathfrak{c}$ of any hypothesis h_i.

Let us write $\mathfrak{c}(\text{'}X \text{ is } B\text{'}, \text{'}X \text{ is } A\text{'}\cdot c) = \mathfrak{c}_B$ and $\mathfrak{c}(\text{'}X \text{ is } \bar{A}\text{'}, \text{'}X \text{ is } \bar{B}\text{'}\cdot c) = \mathfrak{c}_{\bar{A}}$. Thus we have:

(2)
$$\mathfrak{c}_B = \sum_{i=0}^{m} \mathfrak{c}_i\frac{i}{m},$$

and

(3)
$$\mathfrak{c}_{\bar{A}} = \sum_{i=0}^{m} \mathfrak{c}_i\frac{n-m+i}{n} = \sum_{i=0}^{m} \mathfrak{c}_i\frac{n-m}{n} + \sum_{i=0}^{m} \mathfrak{c}_i\frac{i}{n}$$
$$= \frac{n-m}{n} + \mathfrak{c}_B\frac{m}{n}, \quad \text{since } \sum_{i=0}^{m} \mathfrak{c}_i = 1.$$

Hence $\mathfrak{c}_{\bar{A}} - \mathfrak{c}_B = \frac{n-m}{n} + \mathfrak{c}_B\frac{m}{n} - \mathfrak{c}_B = \left(1 - \frac{m}{n}\right)\cdot(1 - \mathfrak{c}_B) > 0$ only if $\mathfrak{c}_B < 1$,

and $\mathfrak{c}_{\bar{A}} - \mathfrak{c}_B$ grows with the growth of $\frac{n}{m}$, q.e.d.

(ii) The $\mathfrak{c}$ that the class of (kitchen) salts is *homogeneous* as far as solubility in water is concerned is high and much greater than the $\mathfrak{c}$ that the class of substances which are not soluble in water is homogeneous with respect to their not being (kitchen) salts.

I shall say that a class A is *homogeneous* with respect to B, if either every A is B or every A is $\bar{B}$, i.e. if one A is B, every A is B, and if one A is $\bar{B}$, no A is B.

Now $\mathfrak{c}$('If one sample of (kitchen) salt is soluble in water, every sample of (kitchen) salt is soluble in water', c) is much greater[7] than $\mathfrak{c}$('If one substance

[7] As a matter of fact the difference between the extensions of the class of substances insoluble in water and of the class of salts alone makes it equal or greater. For the $\mathfrak{c}$ of the class A being homogeneous with respect to B is equal to or greater than the $\mathfrak{c}$ of the class $\bar{B}$ being homogeneous with respect to A—if the extension of $\bar{B}$ is greater than that of A; and it is greater if $\mathfrak{c}$ ('No $\bar{A}$ is $\bar{B}$', c) $\neq 0$.

In fact, $\mathfrak{c}$ ('Every or no A is B', c) $=$ $\mathfrak{c}$ ('Every A is B', c) $+$ $\mathfrak{c}$ ('No A is B', c), $\mathfrak{c}$('No or every $\bar{B}$ is A', c) $=$ $\mathfrak{c}$ ('No $\bar{B}$ is A', c) $+$ $\mathfrak{c}$ ('Every $\bar{B}$ is A', c). The first members of the sums are equal, since 'Every A is B' and 'No $\bar{B}$ is A' are equivalent propositions. (It can be easily shown on the basis of the above axioms that equivalent sentences have the same $\mathfrak{c}$.) But $\mathfrak{c}$ ('Every $\bar{B}$ is A', c) $= 0$, because there are more $\bar{B}$'s than A's.

But the difference between the $\mathfrak{c}$'s of the two homogeneous classes (that of the class A with respect to B and that of the class $\bar{B}$ with respect to A) may be great or small independently of the difference between the extensions of $\bar{B}$ and A, unless this last difference is 0.

insoluble in water is (kitchen) salt, every substance insoluble in water is (kitchen) salt', c); or at least the first $\mathfrak{c}$ *was* much greater than the second when people first began to examine salts.[8]

If we agree with this we have another reason for the fact that instance a_1' raises the $\mathfrak{c}$ of b_1' and therefore of b' by a smaller amount than does instance a'. For it can be proved (see next demonstration) that, with a given $\mathfrak{c}$('Every A is B', c), the greater is the difference between the $\mathfrak{c}$'s for the two homogeneities considered, the greater is the difference, *ceteris paribus*, between the $\mathfrak{c}$ of a $\bar{B}$ being $\bar{A}$ and the $\mathfrak{c}$ of an A being B.[9] Thus, using f_4), the difference between the increase of the $\mathfrak{c}$ of the generalization 'Every A is B' under the influence of its particular case and the increase of the $\mathfrak{c}$ of this generalization under the influence of a particular case of its contrapositive 'Every $\bar{B}$ is $\bar{A}$,' becomes great.

It will also be proved (in the next demonstration) that if the $\mathfrak{c}$ for the homogeneity of A with respect to B tends to 1, the ratio of the $\mathfrak{c}$ of A being B to the $\mathfrak{c}$ of the generalization 'Every A is B' also tends to 1. But this ratio, as it is seen from (I), presents the $\mathfrak{c}$ of the generalization under the influence of one of its observed instances. Thus, if the $\mathfrak{c}$ for the homogeneity of A with respect to B tends to 1, the $\mathfrak{c}$ obtained by the generalization from an observed instance of it also tends to 1; whereas, when the extension of $\bar{B}$ is greater than that of $\bar{A}$, an observed instance for the contrapositive of the generalization leaves the $\mathfrak{c}$ of the generalization far from 1 if the ratio of the $\mathfrak{c}$ for the considered homogeneities is great.

Moreover, it seems that the $\mathfrak{c}$ for the homogeneity of the class of (kitchen) salts with respect to solubility in water is high in itself. We thus have still another ground for the difference in the increase of the $\mathfrak{c}$ of the generalization by its instance and the increase of the $\mathfrak{c}$ of the generalization by an instance of its contrapositive.[10]

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[8] At present the $\mathfrak{c}$ of the generalization 'Every kitchen salt is soluble in water' is so high that not only an instance of its contrapositive but also an instance of itself raises its $\mathfrak{c}$ by a minute value only, because the $\mathfrak{c}$ of this instance is nearly 1. (See f_4), but also f_5), p. 134). Moreover, the $\mathfrak{c}$ of the contrary proposition 'No kitchen salt is soluble in water' is at the present state of knowledge near 0, so that the difference between the two homogeneities considered in (ii) is small (see footnote 7). We understand by 'no kitchen salt' 'no kitchen salt not yet examined,' for otherwise the $\mathfrak{c}$ of the sentence 'no kitchen salt is soluble in water' would be 0.

Instead of the solubility of salt in water, which we took because (kitchen) salt and water are generally known substances, we could consider, e.g., the solubility of boron chloride in diethylene glycol, which is probably not yet examined.

[9] The first $\mathfrak{c}$ being always greater than the second, when $\text{NC}(\bar{B}) > \text{NC}(A)$, as the previous demonstration showed.

[10] To make the notion of 'homogeneity' clearer let us give the two following examples, the first where the $\mathfrak{c}$ of the homogeneity is 1 or nearly 1, the second where it is near 0. Let us place one ball in each of a great number of empty urns, putting in a white or a black one according to the outcome of tossing a coin.

1. We make several drawings from one and the same urn, replacing the ball into the urn after each drawing.

2. We make several drawings, each time from a different urn.

Consider the $\mathfrak{c}$ of the sentence b_2: 'Every drawing gives a white ball.' In the first case supposed, one drawing determines the contents of the urn, i.e. makes the $\mathfrak{c}$ of b_2 equal 1, independently of the extension of its subject term.

Demonstration: Let us use the same notation and the same assumptions a), b), and c) as in the previous demonstration.

We had there $c_{\bar{A}} - c_B = \left(1 - \frac{m}{n}\right)(1 - c_B)$.

1. Let us assume now that c_0, which constitutes the difference between the c's for the two homogeneities considered (see footnote 7) increases, that c_m does not change, and that the c_i, for $i = 1, 2, \cdots, m - 1$ decrease in equal proportion. Then, it is easy to see, (2) may be written:

$$(2') \qquad c_B = c_m + \sum_{i=1}^{m-1} c_i \frac{i}{m} + c_0 \cdot 0,$$

so that c_B decreases. Thus $c_{\bar{A}} - c_B$ increases, q.e.d.

2. Let us suppose that $c_m + c_0 = h$ and that h varies and tends to 1. Then $\sum_{i=1}^{m-1} c_i$ tends to 0 and $\frac{c_m}{c_b}$ tends also to 1, because of (2). If $h = 1$, $\frac{c_m}{c_B} = 1$.

But since

$$(3) \qquad c_{\bar{A}} = c_m + \sum_{i=1}^{m-1} c_i \frac{n - m + i}{n} + c_0 \frac{n - m}{n},$$

the ratio of $c_{\bar{A}}$ to $c_m + c_0 \frac{n - m}{n}$ tends to 1, since $\sum_{i=1}^{n-1} c_i \frac{n - m + i}{n}$ tends to 0.

Thus the ratio of $c_{\bar{A}}$ to $\dfrac{c_m}{c_m + c_0 \frac{n - m}{n}}$ or $\dfrac{1}{1 + \frac{c_0}{c_m} \frac{n - m}{n}}$ also tends to 1. But this last quotient is far from 1, if $\frac{c_0}{c_m}$ is large. Q.e.d.

Now, if the above explanation of Hempel's paradox is right, the paradox should disappear when we choose an example of a sentence 'Every A is B' such that the extension of A is greater than that of $\bar{B}$,[11] the c of $\bar{B}$ being homogeneous with respect to $\bar{A}$ is greater than the c of A being homogeneous with respect to B, and such that $\bar{B}$ being $\bar{A}$ is not certain (or almost certain) a priori. Such an example is obtained by simply taking b_1', i.e., 'Every substance which is not soluble in water is not (kitchen) salt,' as 'Every A is B'; b', i.e., 'Every (kitchen) salt dissolves in water,' as its contrapositive; and a'—a sample of salt which dissolves in water—as an instance of b'. This example should raise the c of b' and thus also of b_1', i.e., of the sentence 'No substance which is insoluble in water is salt.'

This example seems to me to yield no greater paradox than one may legitimately expect. This conclusion is all the more persuasive if we note that after

[11] As a matter of fact the great majority of general sentences which we may think of at random are such that their subject term is less extensive than the negation of their predicate term. And it is so, because, in most cases, the negations of names which are in use are more extensive than the names themselves. Thus, if we think of a general sentence 'Every A is B' at random, without artificially constructing it, there is a large chance that $\bar{B}$ will be more extensive than B and a fortiori than A.

having exhausted all or almost all $\bar{B}$'s, i.e. all or almost all samples of salt, and after having found that they are soluble in water, i.e. after having gained the c of b' equal to 1 or very near 1 (it is much easier to do this by examining instances a' than by examining instances a_1') we obtain certainty or almost certainty also with respect to b_1'.

Should, however, somebody still feel a paradox even here, this may be so because the equivalence of b' and b_1' is forgotten when one thinks about the confirmation of b_1. If in spite of noting this equivalence the paradox still remains, nothing can be done but to give up the principle that facts confirming a sentence also confirm its equivalent. But this would, I think, violate the meaning of confirmation. It would also contradict the above axioms.[12]

5. f_8) It is often maintained that the characteristic of a "metaphysical" theory is that it does not imply any observable fact.

But, as a matter of fact, "metaphysical" theories, if they are not senseless sentences, may very well imply observable facts. Thus the hypothesis that a hidden entity (substance or force or the like), called entelechy or otherwise, causes the turning of plants towards light, implies the observable facts of the turning of plants towards light. And in general, for every set of observable facts α we may offer a metaphysical assumption about the existence of a b, add to it the further metaphysical assumption that every time b occurs, it produces a fact belonging to α, and take both assumptions as a hypothesis b_1. Such hypothesis obviously implies the observable facts belonging to α. α may also contain facts not yet observed; and subsequent observations may then either negate those facts and thus falsify the hypothesis, or exhibit those facts and not falsify the hypothesis.

Consequently, the distinctive feature of hypotheses called metaphysical (i.e. that they do not imply observable facts) must, I think, be specified in terms of c.

The following is one possible way of doing this, which involves the use of f_5):

The a priori c of such a hypothesis b_1 is so small that only very many observed strong[3] facts could raise it, i.e. approach it to certainty by a perceptible amount. Moreover, there is a small a priori c, in the present state of our knowledge, that such facts could be tested, i.e. their occurrence or non-occurrence observed.

As to the facts α implied by b_1, which are explicitly implied by b, their occurrence does not raise the c of b_1 perceptibly; their a priori c is great, since they also follow from well confirmed hypotheses. Thus, to take the hypothesis about entelechies as an example, there is a hypothesis about the chemical constitution of plants and its influence on the behavior of plants in daylight, which implies the turning of plants towards light and is well confirmed a priori.

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[12] In an "existential" sense of the proposition 'Every A is B,' this proposition presupposes the existence of A. Thus 'Every A is B' is not equivalent to 'Every $\bar{B}$ is $\bar{A}$' which presupposes in its turn the existence of $\bar{B}$, but does not presuppose the existence of A.

But the paradox persists when we interpret the proposition 'Every A is B' and its contrapositive in a nonexistential sense. Furthermore, we have assumed the existence of A (and of $\bar{B}$) in our examples prior to the confirmation of the general proposition. Thus, the rôle of a confirming instance is not to establish the existence of A.

The restriction made, "in the present state of our knowledge," seems to me essential. For hypotheses which may at one time be found metaphysical may later become entirely "scientific"[13] after we have gained more knowledge. E.g., the atomic hypothesis was metaphysical in antiquity, but is no longer. Again, if somebody had offered the hypothesis no more than fifty years ago that mutations in living organisms are caused by a kind of rays, not yet "observed," and had called them "cosmic rays," this would have been said to be a metaphysical hypothesis. What has changed since? We now have new observations confirming the existence of "cosmic" rays (the existence of b in our above notation) and new observations confirming their causing mutations (the production of α by b). The hypothesis has simply been confirmed to a greater degree, its $\mathfrak{c}$ has grown. Moreover, the $\mathfrak{c}$ has also grown, that new facts will be tested the occurrence of which would confirm and the non-occurrence disconfirm the hypothesis.

6. By the application of (II) and some other theorems following from Axioms I–IV, among others of a proposition whose immediate particular case is (I), we can also solve the question:

f_9) whether the $\mathfrak{c}$ of a fact a_2 is increased when there is observed another fact a_1 which is a consequence of the same hypothesis b as the first fact but which is not certain a priori.

This question cannot, in general, be answered affirmatively, and a generally affirmative answer would lead to absurdity. For every two facts have a common reason; thus, the facts A_1 is B and C_1 is D, where A_1 is A and C_1 is C, have a common reason 'Every A is B and every C is D.' E.g., 'This piece of iron melts at 1500°' and 'Water in this vessel boils at 100°' have the common reason 'Iron melts at 1500° and water boils at 100°.'

Reichenbach in his *Experience and prediction* (pp. 372–3) advances the following view: Newton discovered a formula which implies the laws of both Galileo and Kepler; in consequence, the observational material supporting Galileo's law also supports the law of Kepler (and vice versa). Now this view may be quite correct; but the question is on what grounds it is to be justified. It has already been seen that in general one consequence of a hypothesis does not support, i.e. raise the $\mathfrak{c}$, of another consequence. However, by applying (II) we can give a sufficient condition for the increase of the $\mathfrak{c}$ of one consequence of b by another of its consequences.

We have in fact the following:

(1) a_1 and a_2 follow from b and c.

(2) $\mathfrak{c}(b, c) \neq 0$, i.e. the $\mathfrak{c}$ of b is not 0 a priori.

(3) $\mathfrak{c}(a_1, c) \neq 1 \neq \mathfrak{c}(a_2, c)$, for otherwise the $\mathfrak{c}$ of a_2 could not increase.

As we saw, these conditions are not sufficient in order that

$$\mathfrak{c}(a_2, a_1 \cdot c) > \mathfrak{c}(a_1, c). \tag{4}$$

[13] There are hypotheses for which it is expressly assumed that they do not imply any observed fact or for which it can be logically demonstrated that no observable fact could confirm or disconfirm them. But these hypotheses do not exhaust those which are called metaphysical.

Now, the following condition (5) together with (1)–(3) implies (4):
(5) $\bar{b}$ disjoins into k mutually exclusive alternatives $b_1, b_2, \cdots, b_k$, with $k \geqq 1$ which confer upon a_2 *the same* c as upon a_1, independently of the observation of a_1, i.e.

$$\mathfrak{c}(a_1, b_i \cdot c) = \mathfrak{c}(a_2, b_i \cdot c) = \mathfrak{c}(a_2, a_1 \cdot b_i \cdot c)$$

for $i = 1, 2, \cdots, k$ with $k \geqq 1$, where b_i implies $\bar{b}_j$ for $j \neq i$ and $\left(\sum_{i=1}^{k} b_i\right) \equiv \bar{b}$.[14]

We could prove this by applying (II); but for the sake of further considerations it will be convenient to prove a more general Theorem 1, needing the application of (II′), whose immediate particular case is (II).

Theorem 1. Let $b_1, b_2, \cdots, b_k$ be concurring hypotheses ($k \geqq 2$) not excluded a priori and whose logical sum is a priori certain. That is,

$$\mathfrak{c}(b_i, c) \neq 0, \tag{6}$$

$$\mathfrak{c}\left(\sum_{i=1}^{k} b_i, c\right) = 1. \tag{7}$$

Furthermore, let $a_1, a_2, \cdots, a_n$ be facts which possess constant c's with respect to each b_i and which are independent of previously observed facts a_i. That is,
(8) $\mathfrak{c}(a_s, b_i \cdot c) = \mathfrak{c}(a_s, b_i \cdot a_1 \cdot a_2 \cdots a_{s-1} \cdot c) = \text{const} \neq 0$ for every i $(1 \leqq i \leqq k)$ and for every s $(1 \leqq s \leqq n)$.
Finally, let
(9) $\mathfrak{c}(a_s, b_i \cdot c) \neq \mathfrak{c}(a_s, b_j \cdot c)$ for at least one pair of i, j such that $i \neq j$. Then

$$\mathfrak{c}(a_n, a_1 \cdot a_2 \cdots a_{n-1} \cdot c) > \mathfrak{c}(a_{n-1}, a_1 \cdot a_2 \cdots a_{n-2} \cdot c). \tag{10}$$

It is easy to see that the statement immediately preceding Theorem 1 is a special case of this theorem, namely when $n = 2$ and $\mathfrak{c}(a_s, b_t \cdot c) = 1$ for some $1 \leqq t \leqq k$ (we must then assume $b \equiv b_t$ and since $\bar{b}$ disjoins into $k \geqq 1$ concurring hypotheses, we have, together with b, $k \geqq 2$ hypotheses, which are required by our theorem.

Demonstration: The following theorem may be obtained from Axioms I–IV:

$$\mathfrak{c}(a, c) = \sum_{i=1}^{k} \mathfrak{c}(b_i, c) \cdot \mathfrak{c}(a, b_i \cdot c) \tag{II′}$$

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[14] Instead of (5) the following condition is also sufficient for (4) when (1)–(3) are assumed:

$$\mathfrak{c}(a_2, \bar{b} \cdot c) \leqq \mathfrak{c}(a_2, a_1 \cdot \bar{b} \cdot c).$$

The demonstration is based upon the two equalities,

$$\mathfrak{c}(a_1, c) = \mathfrak{c}(b, c) + \mathfrak{c}(\bar{b}, c) \cdot \mathfrak{c}(a_1, \bar{b} \cdot c) \text{ and}$$
$$\mathfrak{c}(a_2, a_1 \cdot c) = \mathfrak{c}(b, a_1 \cdot c) + \mathfrak{c}(\bar{b}, a_1 \cdot c) \cdot \mathfrak{c}(a_2, a_1 \cdot \bar{b} \cdot c),$$

which are cases of (II).

A necessary and sufficient condition for (4), when (1), (2) and (3′) are assumed, with (3′) as $\mathfrak{c}(a, c) \neq 1 \neq \mathfrak{c}(a_2, a_1 \cdot c)$, is:

$$\frac{\mathfrak{c}(b, a_1 \cdot c) - \mathfrak{c}(b, c)}{\mathfrak{c}(\bar{b}, c)} > \frac{\mathfrak{c}(a_2, \bar{b} \cdot c) - \mathfrak{c}(a_2, \bar{b} \cdot a_1 \cdot c)}{\mathfrak{c}(\bar{a}_2, \bar{b} \cdot a_1 \cdot c)},$$

as is evident from the above two inequalities.

where b_i are concurring hypotheses whose logical sum is certainty, i.e. $\sum_{i=1}^{k} \mathfrak{c}(b_i, c) = 1$. As is easily seen, (II) is a special case of (II′), when for a certain $b_i \equiv b$, $\mathfrak{c}(a, b \cdot c) = 1$.

From (II′) we have the following two equalities:

$$(11)\ \mathfrak{c}(a_{n-1}, a_1 \cdot a_2 \cdots a_{n-2} \cdot c) = \sum_{i=1}^{k} \mathfrak{c}(b_i, a_1 \cdot a_2 \cdots a_{n-2} \cdot c) \cdot \mathfrak{c}(a_{n-1}, b_i \cdot a_1 \cdot a_2 \cdots a_{n-2} \cdot c) = \sum_{i=1}^{k} \phi_i x_i .$$

$$(12)\ \mathfrak{c}(a_n, a_1 \cdot a_2 \cdots a_{n-1} \cdot c) = \sum_{i=1}^{k} \mathfrak{c}(b_i, a_1 \cdot a_2 \cdots a_{n-1} \cdot c) \cdot \mathfrak{c}(a_n, b_i \cdot a_1 \cdot a_2 \cdots a_{n-1} \cdot c) = \sum_{i=1}^{k} \psi_i x_i .$$

the second factors of both sums being equal by assumption (8).

Let the members of the sum on the right hand side of (11) be ordered with respect to increasing magnitude of x_i, i.e.

$$(13)\qquad x_{i+1} \geqq x_i.$$

We then have

$$(14)\qquad \frac{\psi_{i+1}}{\phi_{i+1}} \geqq \frac{\psi_i}{\phi_i}$$

i.e. the increase of the $\mathfrak{c}$ under the influence of a_{n-1} is equal or greater for those b_i which afford for a_{n-1} an equal or greater $\mathfrak{c}$ respectively.

To see this, we must apply the following proposition which is a consequence of the above axioms:

$$(\mathrm{I}')\qquad \mathfrak{c}(b, a \cdot c) = \frac{\mathfrak{c}(b, c) \cdot \mathfrak{c}(a, b \cdot c)}{\mathfrak{c}(a, c)}, \text{ if } \mathfrak{c}(a, c) \neq 0.$$

We see that (I) is an immediate special case of (I′) if a is a consequence of b and c.

Let us substitute in (I′) a_{n-1} for a, b_i for b and $a_1 \cdot a_2 \cdots a_{n-2} \cdot c$ for c. We have then:

$$(15)\qquad \psi_i = \frac{\phi_i x_i}{\mathfrak{c}(a_{n-1}, a_1 \cdot a_2 \cdots a_{n-2} \cdot c)}.$$

Now, if (13), then by (15) also (14). Hence, we may write instead of (12):

$$\mathfrak{c}(a_n, a_1 \cdot a_2 \cdots a_{n-1} \cdot c) = \sum_{i=1}^{k} (\phi_i + \epsilon_i) \cdot x_i .$$

I.e. by (11) we have:

$$(16)\qquad \mathfrak{c}(a_n, a_1 \cdot a_2 \cdots a_{n-1} \cdot c) = \mathfrak{c}(a_{n-1}, a_1 \cdot a_2 \cdots a_{n-2} \cdot c) + \sum_{i=1}^{k} \epsilon_i \cdot x_i .$$

Now the second member of the right hand side of (16) must be positive. For $\sum_{i=1}^{k} \epsilon_i = 0$, since $\sum_{i=1}^{k} \phi_i = \sum_{i=1}^{k} \psi_i = 1$ because of (7) and f_1). We can thus disjoin $\sum_{i=1}^{k} \epsilon_i \cdot x_i$ into a positive and a negative part, where the positive part is to the right of the negative because of (14). Since the sum of the positive ϵ_i is equal to the absolute value of the sum of the negative ϵ_i, the positive part of $\sum_{i=1}^{k} \epsilon_i x_i$ exceeds the absolute value of the negative part.[15]

Let us now substitute in Theorem 1 '$\bar{b} \cdot c$' for 'c' and take $\left(\sum_{i=1}^{k} b_i\right) \equiv \bar{b}$. (7) is then satisfied, and if (6), (8), and (9) also hold, we have, instead of (10):

$$\text{(10')} \qquad \mathfrak{c}(a_n,\ a_1 \cdot a_2 \cdots a_{n-1} \cdot \bar{b} \cdot c) > \mathfrak{c}(a_{n-1},\ a_1 \cdot a_2 \cdots a_{n-2} \cdot \bar{b} \cdot c).$$

Let us assume further that $\mathfrak{c}(a_s,\ b \cdot c) = 1$ for every $s = 1, 2, \cdots, n$. (10') then says that, assuming $\bar{b}$, i.e. the falsity of our hypothesis b, the $\mathfrak{c}$ of a new instance of b increases with the number of observed instances of it.

Thus, Nicod was right as to this point, in his dispute with Keynes (see *Le problème logique de l'induction*, p. 76)[16] concerning the limit of $\mathfrak{c}(b,\ a_1 \cdot a_2 \cdots a_n \cdot c)$ when n is tending to infinity. For on Keynes' "Principle of Limited Variety," conditions (6), (7), (8), and (9) with $\bar{b} \cdot c$ instead of c are fulfilled, where b is a generalization of $a_1 \cdot a_2 \cdots a_n$.

Nicod was wrong, however, in supposing that $\mathfrak{c}(a_n,\ a_1 \cdot a_2 \cdots a_{n-1} \cdot \bar{b} \cdot c)$ may tend to 1 when n tends to infinity, assuming also Keynes' Principle of Limited Variety, which Nicod finds "fort acceptable." For it can be demonstrated that

$$\text{(17)} \qquad \lim_{n \to \infty} \mathfrak{c}(a_n,\ a_1 \cdot a_2 \cdots a_{n-1} \cdot \bar{b} \cdot c) = \mathfrak{c}(a_s,\ b_t \cdot c)$$

if b_t is that alternative among the alternatives $b_1, b_2, \cdots, b_k$ forming $\bar{b}$ which gives to a_s the maximum $\mathfrak{c}$.

Thus, $\lim_{n \to \infty} \mathfrak{c}(a_n,\ a_1 \cdot a_2 \cdots a_{n-1} \cdot \bar{b} \cdot c)$ could be 1 only by assuming that for some t, $\mathfrak{c}(a_s,\ b_t \cdot c) = 1$; but according to Keynes' Principle of Limited Variety all alternatives into which $\bar{b}$ disjoins give to a_s a $\mathfrak{c}$ smaller than 1.

To prove (17) we shall need the following theorem:

Theorem 2. Let the conditions (6), (7), (8) and (9) of Theorem 1 be fulfilled, for $k \geqq 1$. We may then prove that

$$\text{(18)} \qquad \lim_{n \to \infty} \mathfrak{c}(b_1,\ a_1 \cdot a_2 \cdots a_n \cdot c) = 1, \text{ where}$$

$$\text{(19)} \qquad \mathfrak{c}(a_s,\ b_1 \cdot c) = \max \mathfrak{c}(a_s,\ b_i \cdot c).$$

If there is more than one $\max \mathfrak{c}(a_s,\ b_i \cdot s)$, we may take their logical sum as b_1.

Demonstration: Combining (I') and (II') and substituting 'a_1', 'a_2', $\cdots$, 'a_n' for 'c', we have

$$\text{(20)} \qquad \begin{aligned} \mathfrak{c}(b_1,\ a_1 \cdot a_2 \cdots a_n \cdot c) &= \frac{\mathfrak{c}(b_1,\ c) \cdot \mathfrak{c}(a_1 \cdot a_2 \cdots a_n,\ b_1 \cdot c)}{\sum_{i=1}^{k} \mathfrak{c}(b_i,\ c) \cdot \mathfrak{c}(a_1 \cdot a_2 \cdots a_n,\ b_i \cdot c)} \\ &= \frac{1}{1 + \sum_{i=2}^{k} \dfrac{\mathfrak{c}(b_i,\ c) \cdot \mathfrak{c}(a_1 \cdot a_2 \cdots a_n,\ b_i \cdot c)}{\mathfrak{c}(b_1,\ c) \cdot \mathfrak{c}(a_1 \cdot a_2 \cdots a_n,\ b_1 \cdot c)}}. \end{aligned}$$

[15] There is another demonstration of Theorem 1 by S. Bernstein, ***Theory of probabilities*** (in Russian, pp. 84–85).

[16] The dispute is carried on in terms of probability.

By applying Axiom III several times we get

$$\text{(21)}\quad \mathfrak{c}(a_1\cdot a_2 \cdots a_n, b_i\cdot c) = \mathfrak{c}(a_n, b_i\cdot a_1\cdot a_2 \cdots a_{n-1}\cdot c)\cdot \mathfrak{c}(a_{n-1}, b_i\cdot a_1\cdot a_2 \cdots a_{n-2}\cdot c) \cdots \mathfrak{c}(a_2, a_1\cdot b_i\cdot c)\cdot \mathfrak{c}(a_1, b_i\cdot c)$$

for $i = 1, 2, \cdots, k$. But all n factors are equal because of (8). Thus,

$$\text{(22)}\quad \begin{gathered} \mathfrak{c}(a_1\cdot a_2 \cdots a_n, b_i\cdot c) = [\mathfrak{c}(a_1, b_i\cdot c)]^n \text{ and} \\ \lim_{n\to\infty} \frac{\mathfrak{c}(a_1\cdot a_2 \cdots a_n, b_i\cdot c)}{\mathfrak{c}(a_1\cdot a_2 \cdots a_n, b_1\cdot c)} = \lim_{n\to\infty} \left[\frac{\mathfrak{c}(a_1, b_i\cdot c)}{\mathfrak{c}(a_1, b_1\cdot c)}\right]^n = 0, \end{gathered}$$

because of (19). (20) and (22) give us

$$\text{(23)}\quad \lim_{n\to\infty} \mathfrak{c}(b_1, a_1\cdot a_2 \cdots a_n\cdot c) = 1, \quad \text{q.e.d.}$$[17]

From Theorem 2, (17) immediately follows. For, applying (II′), we have:

$$\text{(24)}\quad \lim_{n\to\infty} \mathfrak{c}(a_n, a_1\cdot a_2 \cdots a_{n-1}\cdot \bar{b}\cdot c) = \lim_{n\to\infty} \sum_{i=1}^{k} \mathfrak{c}(b_i, a_1\cdot a_2 \cdots a_n\cdot \bar{b}\cdot c) \cdot \mathfrak{c}(a_n\cdot a_1\cdot a_2 \cdots a_{n-1}, b_i\cdot \bar{b}\cdot c).$$

From Theorem 2 it follows that $\lim_{n\to\infty} \mathfrak{c}(b_t, a_1\cdot a_2 \cdots a_k\cdot \bar{b}\cdot c) = 1$ by substituting '$\bar{b}\cdot c$' for 'c' and 'b_t' for 'b_1'. Since by (7) $\sum_{i=1}^{k} \mathfrak{c}(b_i, a_1\cdot a_2 \cdots a_n\cdot \bar{b}\cdot c) = 1$ and $\lim_{n\to\infty} \mathfrak{c}(b_i, a_1\cdot a_2 \cdots a_n\cdot \bar{b}\cdot c) = 0$ for $i \neq t$,

$$\begin{aligned} \lim_{n\to\infty} \mathfrak{c}(a_n, a_1\cdot a_2 \cdots a_{n-1}\cdot \bar{b}\cdot c) &= \mathfrak{c}(a_n, a_1\cdot a_2 \cdots a_{n-1}\cdot b_t\cdot \bar{b}\cdot c) \\ &= \mathfrak{c}(a_n, b_t\cdot c), \text{ by Axiom IV and (8),} \end{aligned}$$

because b_t follows from $\bar{b}$ and a_n is independent of $a_1, a_2, \cdots, a_{n-1}$ when assuming b_t, q.e.d.

7. The last question to be discussed is f_{10}), whether the $\mathfrak{c}$ of a hypothesis b, not excluded a priori, tends to 1 when the number of its observed instances $a_1, a_2, \cdots, a_n$, which are not certain before being observed, tends to infinity.

In general, this question must also be answered in the negative.

To show this, combine (I) and (II) and substitute '$a_1\cdot a_2 \cdots a_n$' for 'a', thus obtaining

$$\text{(25)}\quad \mathfrak{c}(b, a_1\cdot a_2 \cdots a_n\cdot c) = \frac{\mathfrak{c}(b, c)}{\mathfrak{c}(b, c) + \sum_{i=1}^{k} \mathfrak{c}(b_i, c)\cdot \mathfrak{c}(a_1\cdot a_2 \cdots a_n, b_i\cdot c)};$$

b_i $(i = 1, 2, \cdots, k)$ are here alternatives into which $\bar{b}$ disjoins, as in (II).

Suppose α) that all facts $a_1, a_2, \cdots, a_n$ which are consequences of b are also consequences of another hypothesis b_t concurring with b and not excluded a priori. Then $\mathfrak{c}(a_1\cdot a_2 \cdots a_n, b_t\cdot c) = 1$, and as may easily be seen from (25)

$$\lim_{n\to\infty} \mathfrak{c}(b, a_1\cdot a_2 \cdots a_n\cdot c) \leq \frac{\mathfrak{c}(b, c)}{\mathfrak{c}(b, c) + \mathfrak{c}(b_t, c)} < 1,$$

[17] See J. Hosiasson, *Quelques remarques sur la dépendance des probabilités a posteriori de celles a priori*, ***Comptes-rendus du I Congrès des Mathématiciens des Pays Slaves***, Warsaw 1930.

since $\mathfrak{c}(b_t, c) \neq 0$, so that the limit of the $\mathfrak{c}$ of b depends on the a priori $\mathfrak{c}$'s of b and b_t.[18]

Suppose β) that $a_1, a_2, \cdots, a_n$ do not follow from another hypothesis concurring with b, but that some of these hypotheses, say hypothesis b_t, not excluded a priori, confers upon the a_i a $\mathfrak{c}$ tending to 1, when the number of observed a_i tends to infinity;[19] that is,

$$\lim_{n\to\infty} \mathfrak{c}(a_n, a_1 \cdot a_2 \cdots a_{n-1} \cdot b_t \cdot c) = 1. \tag{26}$$

But it *may* also be the case that $\lim_{n\to\infty} \mathfrak{c}(b, a_1 \cdot a_2 \cdots a_n \cdot c) < 1$. For from (21) we obtain:

$$\mathfrak{c}(a_1 \cdot a_2 \cdots a_n, b_t \cdot c) = \mathfrak{c}(a_n, a_1 \cdot a_2 \cdots a_{n-1} \cdot b_t \cdot c) \cdot \mathfrak{c}(a_{n-1}, a_1 \cdot a_2 \cdots a_{n-2} \cdot b_t \cdot c) \cdots \mathfrak{c}(a_1, b_t \cdot c).$$

Thus, when (26) is satisfied, it may be that $\lim_{n\to\infty} \mathfrak{c}(a_1 \cdot a_2 \cdots a_n, b_t \cdot c) = a > 0$. In that case, by (25), it is possible that

$$\lim_{n\to\infty} \mathfrak{c}(b, a_1 \cdot a_2 \cdots a_n \cdot c) \leq \frac{\mathfrak{c}(b, c)}{\mathfrak{c}(b, c) + \mathfrak{c}(b_t, c) \cdot a},$$

which is < 1 and dependent upon the a priori $\mathfrak{c}$'s of b and b_t since $\mathfrak{c}(b_t, c) \neq 0$.[20]

It is easy to show that the negation of both α) and β) is a *sufficient* condition for

$$\lim_{n\to\infty} \mathfrak{c}(b, a_1 \cdot a_2 \cdots a_n \cdot c) = 1. \tag{27}$$[21]

For if no b_i confers on a_s a $\mathfrak{c}$ whose limit equals 1, then, using (21), we see that $\lim_{n\to\infty} \mathfrak{c}(a_1 \cdot a_2 \cdots a_n, b_i \cdot c) = 0$, for every $i = 1, 2, \cdots, k$. Thus, by (25) we have

$$\lim_{n\to\infty} \mathfrak{c}(b, a_1 \cdot a_2 \cdots a_n \cdot c) = \frac{\mathfrak{c}(b, c)}{\mathfrak{c}(b, c)} = 1.$$

Hence in the Nicod-Keynes dispute already mentioned, the latter was right concerning the limit of $\mathfrak{c}(b, a_1 \cdot a_2 \cdots a_n \cdot c)$ being 1. For, on the basis of his Principle of Limited Variety, the negation of α) is fulfilled. But (8) is also fulfilled, which excludes the variability of $\mathfrak{c}(a_s, a_1 \cdot a_2 \cdots a_{s-1} \cdot b_i \cdot c)$ with changing s and thus also excludes β).

In an article devoted to this question P. T. Maker attempted to show that (27) is fulfilled (in terms of probability).[22] His reasoning is however unsatisfactory. For although he assumes Keynes's Principle of Limited Variety (he

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[18] See footnote 17.

[19] Note that we did not here assume condition (8) as to the constancy of $\mathfrak{c}(a_s, a_1 \cdot a_2 \cdots a_{s-1} \cdot b_i \cdot c)$.

[20] See J. Hosiasson, *O prawdopodobieństwie hipotez*, ***Przegląd filozoficzny***, vol. 39 (1936).

[21] Thus we may say that the $\mathfrak{c}$ of the logical sum of hypotheses which are not excluded a priori and which confer on the facts $a_1, a_2, \cdots, a_n$ not certain before being observed a $\mathfrak{c}$ equal to 1 in the limit, tends to 1 when the number of a_i tends to infinity.

[22] P. T. Maker, *A proof that pure induction approaches certainty as its limit*, ***Mind***, vol. 42 (1933), pp. 208–212.

includes it in c), he uses it only to show that $\lim_{n\to\infty} \mathfrak{c}(a_n, a_1 \cdot a_2 \cdots a_{n-1} \cdot c) = 1$.[23] He then says: "As the unexamined instances become increasingly probable, approaching certainty as a limit, the generalization approaches the same limit, since the certainty of both the examined and the unexamined instances of the law implies the certainty of the law." However, I see no justification for this reasoning. The most we can say is that, since the $\mathfrak{c}$ of the unexamined instances approaches certainty as a limit, the $\mathfrak{c}$ of a logical product of a given *finite* number of unexamined instances approaches the same limit. But we cannot say that the $\mathfrak{c}$ of *all* unexamined instances approaches this limit. For the hypothesis b_t, "Every *examined* instance of the law confirms (or will confirm) the law, and an instance of the law never observed does not confirm it," is a hypothesis which is not always excluded a priori; moreover it also gives a $\mathfrak{c}$ equal to 1 to all examined instances confirming the law. But b_t is an alternative concurring with the law itself, i.e. with the law: "Every instance of the law confirms it."

The considerations of this paragraph make us also doubt whether Carnap's Definition 16 in *Testability and meaning*[24] realizes what the author intended.

The class C' is there specified only by the condition that it contains an infinite number of independent consequences of S and therefore not necessarily *all* the consequences of S. Hence S may very well be a logical product of two factors: a given law whose consequences are observed sentences of C', and any sentence whatever with arbitrary terms. On Definition 16, however, S would be said to be confirmable.

On the other hand the $\mathfrak{c}$ of S relatively to C' would not be 1, as our considerations show, and may be even very far from 1. This last point may, however, be no objection for Carnap. For by Decision 5b concerning the admittance of S with a certain degree of confirmation, when sentences of C are stated, S is a law of the form '$(x)Q(x)$' and C the class of sentences '$Q(a_1)$', '$Q(a_2)$', $\cdots$; thus C contains all instances of S and not only a part of them as is the case with C'.

Decision 5b, however, at least on some interpretations of it, is open to certain objections. We shall not raise them, however, since the vagueness of the formulation of Decision 5b does not allow one to decide whether one interpretation or another should be taken. Let us make only the following observation. There is no difference for the import of Decision 5b, if we cut it short at the point where "if no other reasons are against this, . . . " begins. This is so because if the $\mathfrak{c}$ of $\bar{C}$ is high, the $\mathfrak{c}$ of C cannot be also high, as $\mathfrak{c}(C, c) + \mathfrak{c}(\bar{C}, c) = 1$. This follows from the above axioms.

In conclusion, we may observe that Axioms I–IV imply many facts about the $\mathfrak{c}$, solve different questions in a definite and precise way and simplify some statements. They enable us to avoid occasional appeals to intuition, since if they are once accepted, all further facts may be deduced in quite a formal way. Therefore we find them useful as general laws governing degrees of confirmation.

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[23] To show this, Keynes' Principle is not needed. See Poirier, ***Remarques sur la probabilité des inductions,*** Paris 1931, p. 31.

[24] ***Philosophy of science,*** vol. 3 (1936).