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A Solitaire Game and Its Relation to a Finite Field

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The game we are about to discuss is played by one person who uses a board with 33 holes in it as pictured in Figure 1. In the initial position there is a peg in all but one of the holes in the board; usually, the empty hole is in the center of the board. The object of the puzzle is to obtain, by means of a sequence of moves, a position on which only one peg remains on the board. Sometimes it is required that the remaining peg should be in the hole in the center of the board. There is only one type of move, and it consists of jumping one peg over another into an empty hole, and at the same time the peg that has been jumped over is removed from the board. Jumping is allowed both horizontally and vertically, so in every move three consecutive holes in a row of holes in the board are involved. Indicating an occupied hole by x and an empty one by o, the move can be made from xxo to xoo, or from oxx to xoo, or from $\begin{matrix}x\\x\\o\end{matrix}$ to $\begin{matrix}o\\o\\x\end{matrix}$, or from $\begin{matrix}o\\x\\x\end{matrix}$ to $\begin{matrix}x\\o\\o\end{matrix}$.

An extensive description of this game, along with some variations and its history are given in chapter 8 of W. Ahrens' *Mathematische Unterhaltungen und Spiele*, Volume 1 (Leipzig and Berlin, Druck und Verlag von B. G. Teubner, 1910). Quite a lot of attention is given in this chapter to the question of which final positions can be obtained from a given starting position. These considerations can be simplified considerably, both in the formulation of the results or in their derivation, by means of the finite field $GF(4)$ having four elements. Everything we need to know about $GF(4)$ is presented in this note.

The finite field $GF(4)$ contains four elements which we denote by 0, 1, p, and q. The symbols 0 and 1 are used because these elements act as the zero and unit elements. Addition and multiplication of the elements of $GF(4)$ is carried out according to the following tables:

+	0	1	p	q
0	0	1	p	q
1	1	0	q	p
p	p	q	0	1
q	q	p	1	0

addition

×	0	1	p	q
0	0	0	0	0
1	0	1	p	q
p	0	p	q	1
q	0	q	1	p

multiplication

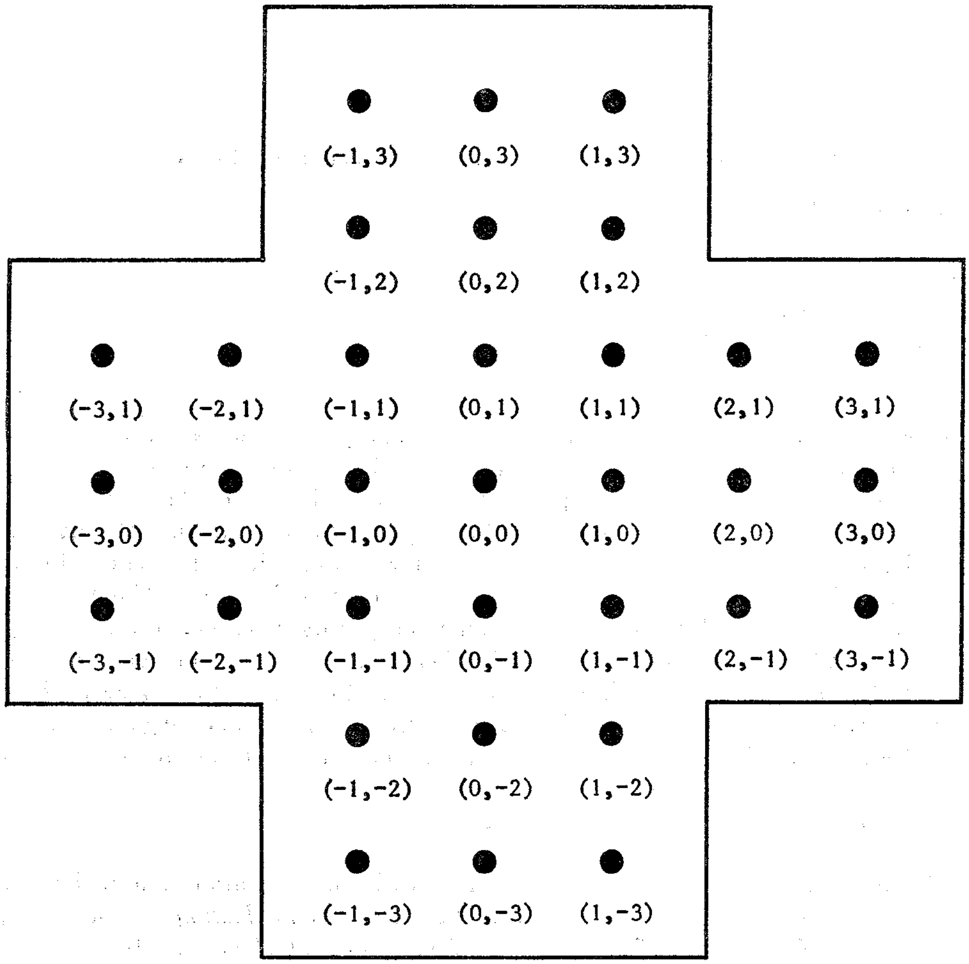

FIGURE 1.
The board used in the solitaire game with its co-ordinate system.

The reader may find it entertaining to verify that these tables imply the following relations:

$$1 + p = p^2, \qquad p + p^2 = 1. \tag{1}$$

Actually, the field $GF(4)$ has been chosen because of the fact that it contains an element p satisfying the relations (1). We are going to show how these relations are connected to the solitaire moves.

To make this connection we introduce a coordinate system on the board. The center of the board is given coordinates $(0,0)$, and the rest of the holes are given coordinates just as though they were integer points in the ordinary Cartesian plane (see Figure 1). By the way, the fact that the board has this particular shape plays no essential role in our considerations. It is even possible to consider the game in more than two dimensions.

Any set of pegs on the board is called a *situation*. If S is a situation, we form the sum

$$A(S) = \sum_{(k,l)\in S} p^{k+l}, \tag{2}$$

where all powers (also powers with negative exponent) are carried out in $GF(4)$, as well as the addition. We give an example. Let there be 5 pegs on the board, at

$$(-1,1), (0,2), (0,-2), (1,1), (2,1), (3,2) \tag{3}$$

then the value of $A(S)$ is

$$p^{-1+1} + p^{0+2} + p^{0-2} + p^{1+1} + p^{2+1} + p^{3+2} = 1 + q + p + q + 1 + q = 1. \tag{4}$$

The expression (2) has the following feature: If T arises from S by a single move, we have $A(S) = A(T)$. That is, $A(S)$ *is constant during the game*. Let us consider a move to the right. It replaces the pegs

$$(k, l) \quad \text{and} \quad (k + 1, l)$$

by a single peg $(k + 2, l)$. The contribution of these pegs to the old situation was $p^{k+l} + p^{k+1+l}$, and to the new situation it is just p^{k+2+l}. And indeed, by (1),

$$p^{k+l} + p^{k+1+l} = p^{k+l}(1 + p) = p^{k+l}p^2 = p^{k+2+l}.$$

The second equation of (1) serves to show that $A(S)$ does not change with moves to the left. Upward moves have the same effect as moves to the right (interchange the roles of k and l), and downward moves are treated as moves to the left.

There is a second expression, similar to $A(S)$, namely,

$$B(S) = \sum_{(k,l)\in S} p^{k-l}, \tag{5}$$

and it can be shown, again by (1), that $B(S)$ is also constant during the game. In the situation (3) it turns out to have the value

$$p^{-1-1} + p^{0-2} + p^{0+2} + p^{1-1} + p^{2-1} + p^{3-2} = p + p + q + 1 + p + p = p.$$

Thus, we have attached to every situation a pair $(A(S), B(S))$ of elements of $GF(4)$. There are 16 different pairs (x, y) with $x \in GF(4)$, $y \in GF(4)$. We can show by examples (see Figure 2) that each one of these can really occur as value of a pair $(A(S), B(S))$.

So the solitaire positions fall into 16 non-empty classes. During a game we stay in one and the same class. If we start the game with all holes filled except the center, we have $A(S) = B(S) = 1$. (The reader may check this by noting that any group of three consecutive pegs gives contribution 0 to both A and B.) It is now easily derived that if we ever succeed in ending with a single peg on the board, it will be at either $(0,0)$, or $(0,\pm 3)$, or $(\pm 3,0)$.

Note that

$$p^3 = q^3 = 1, \qquad 1 + 1 = 0.$$

Therefore, two situations belong to the same class if the one is obtained from the other by shifting a peg three places to the right (or in any of the other 3 directions), or by reducing *xoox* to *oooo* or by reducing *xxx* to *ooo*.

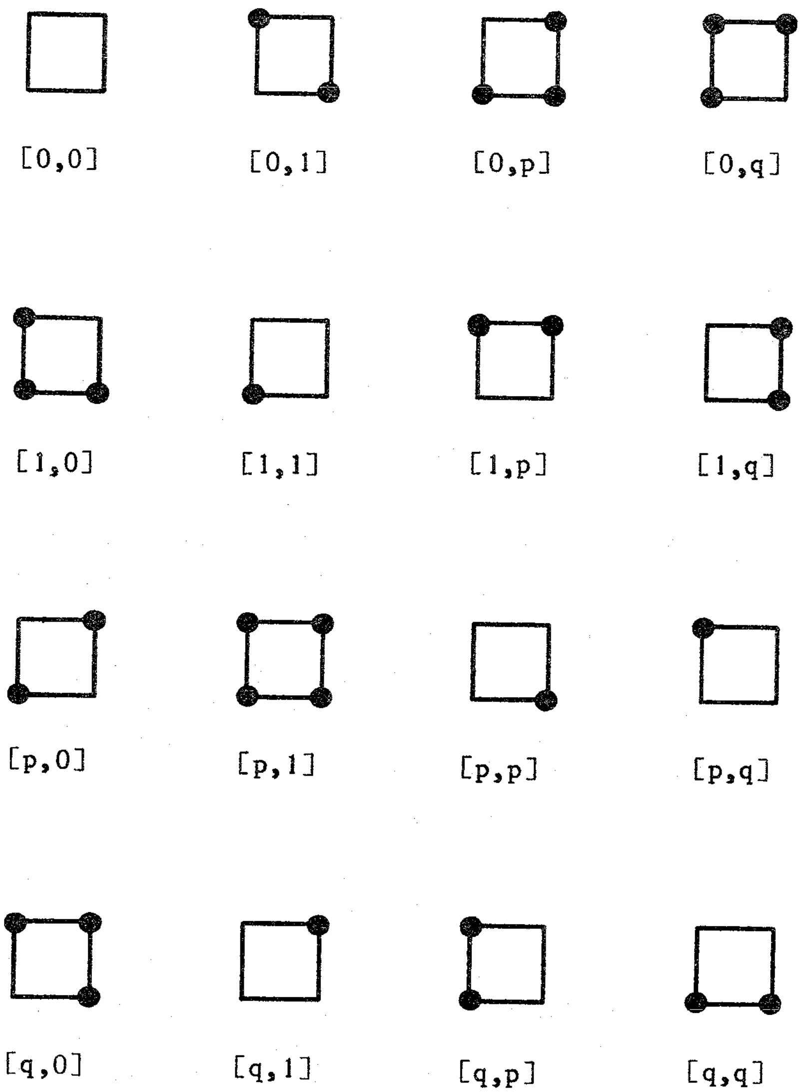

FIGURE 2.

In each one of the 16 figures, the lower left corner represents the center of the board, the other points being (1, 0), (0, 1), (1, 1). Heavy dots represent pegs. The values between square brackets are the values of $A(S)$ and $B(S)$.

If two situations belong to the same class, there is not always a sequence of moves leading from one to the other. For example, it is not hard to construct a non-empty situation with $A(S) = B(S) = 0$, but it is impossible to turn that situation into the empty board by legal play.

The solitaire puzzle is an intelligent and stimulating pastime, provided that one does not memorize sequences of moves. One can play pretty well at random until one has about twelve pegs left. At that point one tries to solve a puzzle, without touching the pegs before a solution is found. In order to have better chances, the puzzler has to take care that the twelve pegs he starts his serious puzzling with, are kept reasonably together. And there may be other little things to keep in mind. For example, if we want to end up with the single peg at $(0,0)$, we should not kill *all* five pegs $(0,0)$, $(0,2)$, $(0,-2)$, $(2,0)$, $(-2,0)$, for no other peg can ever jump into $(0,0)$.

Biographical Sketch

N. G. de Bruijn was born 9 July 1918 in The Hague, The Netherlands. He was one of very few Dutch mathematicians able to finish his Ph.D. degree while his homeland was occupied by the Germans. Shortly after the war, he was appointed Professor of Mathematics in Delft. In 1952 he moved to Amsterdam, and then to Eindhoven in 1960. Besides his contributions to such diverse fields as number theory, asymptotic analysis, combinatorial theory, Banach algebra, and computer languages, de Bruijn has been interested in recreational mathematics all his life. Among his hobbies are collecting scientific toys, and oil painting.

Show me a youngster who has lost a trio of front teeth and I'll show you a three-space.—*Charles W. Trigg*