

# The Continuum Hypothesis, Part I

W. Hugh Woodin

## Introduction

Arguably the most famous formally unsolvable problem of mathematics is Hilbert's first problem:

**Cantor's Continuum Hypothesis:** Suppose that  $X \subseteq \mathbb{R}$  is an uncountable set. Then there exists a bijection  $\pi : X \rightarrow \mathbb{R}$ .

This problem belongs to an ever-increasing list of problems known to be unsolvable from the (usual) axioms of set theory.

However, some of these problems *have* now been solved. But what does this actually mean? Could the Continuum Hypothesis be similarly solved? These questions are the subject of this article, and the discussion will involve ingredients from many of the current areas of set theoretical investigation. Most notably, both *Large Cardinal Axioms* and *Determinacy Axioms* play central roles. For the problem of the Continuum Hypothesis, I shall focus on one specific approach which has developed over the last few years. This should not be misinterpreted as a claim that this is the only approach or even that it is the best approach. However, it does illustrate how the various, quite distinct, lines of investigation in modern set theory can collectively yield new, potentially fundamental, insights into questions as basic as that of the Continuum Hypothesis.

The generally accepted axioms for set theory—but I would call these the twentieth-century choice—are the Zermelo-Fraenkel Axioms together with the Axiom of Choice, ZFC. For a discussion

of these axioms and related issues, see [Kanamori, 1994].

The *independence* of a proposition  $\phi$  from the axioms of set theory is the arithmetic statement: ZFC does not prove  $\phi$ , and ZFC does not prove  $\neg\phi$ .

Of course, if ZFC is *inconsistent*, then ZFC proves anything, so independence can be established only by assuming at the very least that ZFC is consistent. Sometimes, as we shall see, even stronger assumptions are necessary.

The first result concerning the Continuum Hypothesis, CH, was obtained by Gödel.

**Theorem (Gödel).** Assume ZFC is consistent. Then so is ZFC + CH.  $\square$

The modern era of set theory began with Cohen's discovery of the method of *forcing* and his application of this new method to show:

**Theorem (Cohen).** Assume ZFC is consistent. Then so is ZFC + "CH is false".  $\square$

I briefly discuss the methodology for establishing that a proposition is unsolvable, reviewing some basic notions from mathematical logic. It is customary to work within set theory, though ultimately the theorems, fundamentally arithmetic statements, can be proved in number theory.

$\mathcal{L}(\hat{=}, \hat{\in})$  denotes the formal language for set theory; this is a countable collection of *formulas*. The formulas of  $\mathcal{L}(\hat{=}, \hat{\in})$  with no unquantified occurrences of variables are sentences. Both  $\hat{=}$  and  $\hat{\in}$  are simply *symbols* of this formal language with no other a priori significance.

From elementary logic one has the notion of a *structure* for this language. This is a pair  $\mathcal{M} = \langle M, E \rangle$ , where  $M$  is a nonempty set and  $E \subseteq M \times M$  is a binary relation on the set  $M$ . If  $\phi$  is a sentence in the language  $\mathcal{L}(\hat{=}, \hat{\in})$ , then the structure  $\mathcal{M}$  is a *model* of  $\phi$ , written " $\mathcal{M} \models \phi$ " if

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the sentence is true when interpreted as an assertion within the structure  $\langle M, E \rangle$ , where the symbol  $\hat{=}$  is interpreted by the binary relation  $E$  and  $\hat{=}$  is interpreted by equality on  $M$ . Of course, one could consider structures of the form  $\langle M, E, \sim \rangle$ , where  $\sim$  is an equivalence relation on  $M$  intended as the interpretation of  $\hat{=}$ , but then one could pass to the quotient structure  $\langle M/\sim, E/\sim \rangle$ . So nothing is really gained by this attempt at generality.

A *theory* is a set of sentences, and for a given theory  $T$ , I write “ $\langle M, E \rangle \models T$ ” to indicate that  $\langle M, E \rangle \models \phi$  for each sentence  $\phi \in T$ .

ZFC denotes a specific (infinite) theory. A model of ZFC is simply a structure  $\langle M, E \rangle$  such that

$$\langle M, E \rangle \models \text{ZFC}.$$

This can be defined quite naturally without recourse to formal logic. For example, one of the axioms of ZFC is the *Axiom of Extensionality*, which is formally expressible as

$$\forall x_1 \forall x_2 (x_1 \hat{=} x_2 \leftrightarrow \forall x_3 (x_3 \hat{\in} x_1 \leftrightarrow x_3 \hat{\in} x_2))$$

and which informally is just the assertion that two sets are equal if they have the same elements.

Thus

$$\langle M, E \rangle \models \text{“Axiom of Extensionality”}$$

if and only if for all  $a \in M$  and for all  $b \in M$ , if

$$\{c \in M \mid (c, a) \in E\} = \{c \in M \mid (c, b) \in E\},$$

then  $a = b$ .

Therefore  $\langle \mathbb{R}, < \rangle \models \text{“Axiom of Extensionality”}$ ; but if we define  $E \subset \mathbb{N} \times \mathbb{N}$  by

$$E = \{(n, m) \mid \text{For some prime } p, \\ p^{n+1} \mid m \text{ and } p^{n+2} \nmid m\},$$

then  $\langle \mathbb{N}, E \rangle \not\models \text{“Axiom of Extensionality”}$ .

Continuing by examining the remaining axioms, one can develop naturally the notion of a model of ZFC.

One can fairly easily define a model which satisfies all of the axioms of ZFC except for the crucial Axiom of Infinity. For example, let  $E_0$  denote the binary relation on  $\mathbb{N}$  just specified, and define an equivalence relation  $\sim_0$  by  $i \sim_0 j$  if  $\{k \mid (k, i) \in E_0\} = \{k \mid (k, j) \in E_0\}$ . Define a binary relation  $E_1$  by  $(i, k) \in E_1$  if  $(j, k) \in E_0$  for some  $j \sim_0 i$ , and define  $\sim_1$  from  $E_1$  as  $\sim_0$  was defined from  $E_0$ . Continue by induction to define increasing sequences  $\langle \sim_n : n \in \mathbb{N} \rangle$  and  $\langle E_n : n \in \mathbb{N} \rangle$ . Let

$$\sim_\infty = \cup \{\sim_n \mid n \in \mathbb{N}\},$$

and let  $E_\infty = \cup \{E_n \mid n \in \mathbb{N}\}$ . The quotient structure  $\langle \mathbb{N}/\sim_\infty, E_\infty/\sim_\infty \rangle$  satisfies all of the axioms of ZFC except the Axiom of Infinity. The Axiom of Infinity fails to hold, since for each  $i \in \mathbb{N}$  the set of equivalence classes  $\{[j]_{\sim_\infty} \mid (j, i) \in E_\infty\}$  is finite,

for it is equal to the set  $\{[j]_{\sim_\infty} \mid (j, i) \in E_0\}$ , which evidently has cardinality at most  $i$ .

### Building New Models of ZFC

Is there a model of ZFC? A consequence of Gödel’s Second Incompleteness Theorem is that one cannot hope to *prove* the existence of a model of ZFC working just from the axioms of ZFC. Nevertheless, one can still study the problem of building new (hopefully interesting) models of ZFC from given models of ZFC.

Gödel solved the *substructure* problem in 1938, showing that if  $\langle M, E \rangle \models \text{ZFC}$ , then there exists  $M^* \subseteq M$  such that

$$\langle M^*, E \cap (M^* \times M^*) \rangle \models \text{ZFC} + \text{CH}.$$

Over 25 years later Cohen, arguably the Galois of set theory, solved the *extension* problem. The weakest version of Cohen’s extension theorem is actually formally equivalent to the statement of Cohen’s theorem given at the beginning of this article. This weak version simply asserts that if  $\langle M, E \rangle \models \text{ZFC}$ , then there exists a structure

$$\langle M^{**}, E^{**} \rangle \models \text{ZFC} + \text{“CH is false”},$$

such that  $M \subseteq M^{**}$  and such that  $E = E^{**} \cap (M \times M)$ .

Cohen’s method has turned out to be quite powerful: it and its generalizations are the basic tools for establishing independence. Moreover, essentially no other effective method for building extensions of models of ZFC is known.

An important point is that neither Cohen’s method of extension nor Gödel’s method of restriction affects the arithmetic statements true in the structures, so the intuition of a *true* model of number theory remains unchallenged.

It seems that most mathematicians do believe that arithmetic statements are either true or false. No generalization of Cohen’s method has yet been discovered to challenge this view. But this is not to say that such a generalization will never be found.

The empirical completeness of arithmetic coupled with an obvious failure of completeness for set theory has led some to speculate that the phenomenon of independence is fundamental, in particular that the continuum problem is *inherently vague* with no solution. It is in this view a question that is fundamentally devoid of meaning, analogous to asking, “What is the color of  $\pi$ ?”

Wherein lies the truth? I shall begin by describing some classical questions of Second Order Number Theory—this is the theory of the integers together with all sets of integers—which are also not solvable from ZFC. Here I maintain there is a solution: there *are* axioms for Second Order Number Theory which provide a theory as canonical as that of number theory. These relatively new axioms provide insights to Second Order Number Theory which transcend those provided by even ZFC.

Can these axioms be extended to more complicated sets in order to solve the continuum problem? This question will be the focus of the second part of this article.

### Some Preliminaries

For purposes of this article it is convenient to work within set theory. This can initially be conceptually confusing, for we shall work within set theory attempting to analyze set theory.

So let us assume that the universe of sets exists and that the axioms represent true assertions about this universe. We shall initially assume just the axioms of ZFC. Eventually we shall augment these axioms by some modest large cardinal axioms. The discussion to take place simply refers to objects in this universe.

**Definition.** A set  $X$  is *transitive* if each element of  $X$  is also a subset of  $X$ . The *transitive closure* of a set  $X$  is the set  $\bigcap \{Y \mid Y \text{ is transitive and } X \subseteq Y\}$ .  $\square$

Suppose that  $\langle M, E \rangle$  is a model of ZFC. Then the model  $\langle M, E \rangle$  is *transitive* if  $M$  is a transitive set and

$$E = \{(a, b) \mid a \in M, b \in M, \text{ and } a \in b\},$$

so that  $\hat{=}$  is interpreted by actual set membership. Transitive models of ZFC are particularly nice, but they are even harder to come by than arbitrary models. The existence of a model of ZFC does not imply the existence of a transitive model of ZFC.

The theorems of Cohen and Gödel on perturbing models of ZFC are best understood when the initial structure  $\langle M, E \rangle$  is transitive and  $M$  is countable. In the case that  $\langle M, E \rangle$  is transitive, the structure  $\langle M^*, E^* \rangle$  produced by Gödel's construction is again transitive. If  $\langle M, E \rangle$  is transitive and  $M$  is countable, then the structures  $\langle M^{**}, E^{**} \rangle$  produced by Cohen's method can, without loss of generality, be assumed to also be transitive.

The ordinals are those sets  $X$  which are transitive and totally ordered by the membership relation. Thus a transitive set  $X$  is an ordinal if and only if for all  $a \in X$  and for all  $b \in X$ , if  $a \neq b$ , then either  $a \in b$  or  $b \in a$ . A consequence of the axioms is that if  $(L, <)$  is a totally ordered set which is *well ordered* (every (nonempty) subset of  $L$  has a  $<$ -least element), then there is an ordinal  $X$  such that the total orders  $(L, <)$  and  $(X, \in)$  are isomorphic. Collectively the ordinals are well ordered by the membership relation, and this ordering is exactly the ordering arising from the comparison of the order types of well-orders.

The first three ordinals are  $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}$ . The finite ordinals are the nonnegative integers;  $\omega$  denotes the least infinite ordinal, and  $\omega_1$  denotes the least uncountable ordinal. Finally, an ordinal  $\kappa$  is a *cardinal* if there is no bijection of  $\kappa$  with  $\alpha$  for any ordinal  $\alpha < \kappa$ . The finite ordinals are

cardinals, as are  $\omega$  and  $\omega_1$ . The assertion that a set  $X$  has cardinality  $\aleph_1$  is the assertion that there exists a bijection of  $X$  with  $\omega_1$ . Similarly,  $X$  has cardinality  $2^{\aleph_0}$ , or  $c$ , if there is a bijection of  $X$  with  $\mathcal{P}(\mathbb{N})$ , *powerset* of  $\mathbb{N}$ , which is the set of all subsets of  $\mathbb{N}$ .

The ordinals measure height in the universe of sets. Suppose that  $M$  is a transitive set and that  $\langle M, \in \rangle \models \text{ZFC}$ . Then it follows that the set

$$\{a \in M \mid \langle M, \in \rangle \models \text{"}a \text{ is an ordinal"}\}$$

is precisely the set of all ordinals,  $\alpha \in M$ . Moreover, this is an initial segment of the ordinals. Thus the height of  $M$  is precisely the ordinal  $M \cap \text{Ord}$ , where  $\text{Ord}$  denotes the class of all ordinals.

**Definition.** Suppose  $\kappa$  is an infinite cardinal.  $H(\kappa)$  denotes the set of all sets  $X$  whose transitive closure has cardinality less than  $\kappa$ .  $\square$

Every set belongs to  $H(\kappa)$  for sufficiently large cardinals  $\kappa$ . This in the context of the other axioms is equivalent to the Axiom of Choice.

The answer to the continuum problem lies in understanding  $H(\omega_2)$ , where  $\omega_2$  is the smallest cardinal greater than  $\omega_1$ . This suggests an incremental approach. One attempts to understand in turn the structures  $H(\omega)$ ,  $H(\omega_1)$ , and then  $H(\omega_2)$ . A little more precisely, one seeks to find the relevant axioms for these structures. Since the Continuum Hypothesis concerns the structure of  $H(\omega_2)$ , any reasonably complete collection of axioms for  $H(\omega_2)$  will resolve the Continuum Hypothesis.

The first of these structures,  $H(\omega)$ , is a familiar one in disguise:  $\langle \mathbb{N}, +, \cdot \rangle$ . In fact, it can be shown that the structures  $\langle H(\omega), \in \rangle$  and  $\langle \mathbb{N} / \sim_\infty, E_\infty / \sim_\infty \rangle$  are isomorphic, where the latter is defined in the discussion immediately preceding the discussion of building new models of ZFC. Thus number theory is simply set theory in the presence of the negation of the Axiom of Infinity.

The next structure,  $H(\omega_1)$ , is also a familiar one. It is essentially just the structure  $\langle \mathcal{P}(\mathbb{N}), \mathbb{N}, +, \cdot, \in \rangle$ , which is the standard structure for Second Order Number Theory.

Of course, neither  $\langle \mathbb{N}, +, \cdot \rangle$  nor  $\langle \mathcal{P}(\mathbb{N}), \mathbb{N}, +, \cdot, \in \rangle$  is a structure for the language  $\mathcal{L}(\hat{=}, \hat{\in})$ , but each is naturally a structure for a specific formal language which is easily defined.

There are natural questions about  $H(\omega_1)$  which are not solvable from ZFC. However, there are axioms for  $H(\omega_1)$  which resolve these questions, providing a theory as canonical as that of number theory, and which are clearly *true*. But the truth of these axioms became evident only *after* a great deal of work. For me, a remarkable aspect of this is that it demonstrates that the discovery of mathematical truth is not a purely formal endeavor.

The second part of this article will focus on the attempt to find a generalization of these axioms

for  $H(\omega_2)$ . Here is where the answer to the continuum problem lies, for the Continuum Hypothesis is expressible as a proposition about  $H(\omega_2)$ . The surprising answer is that there *are* generalizations but that any generalization which yields a theory which is *strongly* canonical in a certain specific sense must imply that CH is false.

In the course of this discussion I will have to defend the claim that  $\langle H(\omega_2), \in \rangle$ , rather than  $\langle \mathcal{P}(\mathbb{R}), \mathbb{R}, +, \cdot, \in \rangle$ , is the correct immediate generalization of the structure  $\langle H(\omega_1), \in \rangle$  (or, equivalently, of  $\langle \mathcal{P}(\mathbb{N}), \mathbb{N}, +, \cdot, \in \rangle$ ). The structure  $\langle \mathcal{P}(\mathbb{R}), \mathbb{R}, +, \cdot, \in \rangle$ , which is naturally given by the powerset of  $\mathbb{R}$ , has more traditionally been viewed as the next stop on the journey into the transfinite. The method used to analyze the possibilities for strongly canonical theories for  $\langle H(\omega_2), \in \rangle$  actually shows that there can be no strongly canonical theory for  $\langle \mathcal{P}(\mathbb{R}), \mathbb{R}, +, \cdot, \in \rangle$ . If CH holds, then these two structures,  $\langle H(\omega_2), \in \rangle$  and  $\langle \mathcal{P}(\mathbb{R}), \mathbb{R}, +, \cdot, \in \rangle$ , are in essence the same (each can be interpreted in the other), just as are the structures  $\langle H(\omega_1), \in \rangle$  and  $\langle \mathcal{P}(\mathbb{N}), \mathbb{N}, +, \cdot, \in \rangle$ . If CH fails, then these two structures can be very different, with the former structure possibly being fundamentally *simpler* than the latter structure.

### The First Step, $H(\omega_1)$

Consider the following basic operations for subsets of  $\mathbb{R}^n$ ; these generate the *projective sets* from the closed sets.

(Projection) Suppose  $X \subseteq \mathbb{R}^{n+1}$ . The *projection* of  $X$  to  $\mathbb{R}^n$  is the image of  $X$  under the projection map,

$$\pi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n,$$

defined by  $\pi(a_1, \dots, a_n, a_{n+1}) = (a_1, \dots, a_n)$ .

(Complements) Suppose  $X \subseteq \mathbb{R}^n$ . The *complement* of  $X$  is the set

$$X^* = \{(a_1, \dots, a_n) \mid (a_1, \dots, a_n) \notin X\}.$$

**Definition (Luzin).** A set  $X \subseteq \mathbb{R}^n$  is a *projective set* if for some integer  $k$  it can be generated from a closed subset of  $\mathbb{R}^{n+k}$  in finitely many steps, applying the basic operations of taking projections and complements.  $\square$

I caution that because we are working in the Euclidean spaces, it generally requires three applications of our basic operations to reach anything interesting. The projection of a closed subset of  $\mathbb{R}^{n+2}$  yields a subset of  $\mathbb{R}^{n+1}$  which is easily seen to be expressible as a countable union of closed sets. Complementing and projecting again take one beyond the Borel sets and into the *analytic sets*. More formally, a set  $X \subseteq \mathbb{R}^n$  is an analytic set if there exists a closed set  $C \subseteq \mathbb{R}^{n+2}$  such that  $X$  is the projection of  $\mathbb{R}^{n+1} \setminus Y$ , where  $Y$  is the projection of  $C$ .

Why consider the projective sets? The answer is simply that the structure of  $H(\omega_1)$  can be reinterpreted as the structure of the projective sets. More formally, the projective sets correspond to sets  $A \subseteq H(\omega_1)$  for which there is a formula  $\phi(x_1, x_2)$  of  $\mathcal{L}(\hat{=}, \hat{\in})$  and an element  $a \in H(\omega_1)$  such that

$$A = \{b \in H(\omega_1) \mid \langle H(\omega_1), \in \rangle \models \phi[a, b]\}.$$

Such sets  $A$  are *definable*, from parameters, in the structure  $\langle H(\omega_1), \in \rangle$ . This is a common method of logic: study a structure by studying the sets and relations which can be defined in the structure.

The Axiom of Choice implies the existence of many bizarre sets. A well-known example is the Banach-Tarski Paradox: there exists a finite partition of the unit ball of  $\mathbb{R}^3$  into pieces from which two copies of the unit ball can be manufactured using only rigid motions. Such a partition is a *paradoxical partition*.

To guide our discussion, consider the following question.

**Question.** *Can there be a paradoxical partition of the unit ball of  $\mathbb{R}^3$  into pieces, each of which is a projective set?*

Every analytic subset of  $\mathbb{R}^n$  is Lebesgue measurable; this was first proved by Luzin in 1917. As a corollary there can be no paradoxical partition of the unit ball of  $\mathbb{R}^3$  into pieces, each of which is in the  $\sigma$ -algebra generated by analytic sets. This is because any paradoxical partition *must* include pieces which are not Lebesgue measurable.

Of course, our pilot question on projective paradoxical partitions really suggests the more fundamental question:

**Question.** *Are the projective sets Lebesgue measurable?*

By the 1920s the difficulty of this question was apparent:

[Luzin, 1925] one does not know *and one will never know* [of the projective sets whether or not they are each Lebesgue measurable.]

Curiously, Gödel's basic method of showing the (relative) consistency of CH with ZFC yielded a surprising bonus to which Gödel himself attached a significance comparable to that of his results on CH.

**Theorem (Gödel).** *Assume ZFC is consistent. Then so is ZFC + "There is a nonmeasurable projective set".*  $\square$

In fact, an immediate corollary of the proof of this theorem is the (relative) consistency with the axioms of set theory of the statement:

There exists a paradoxical partition of the unit ball of  $\mathbb{R}^3$  into pieces, each of which is the projection of the complement of an analytic set.

So Luzin's theorem on the Lebesgue measurability of the analytic sets was in fact the strongest theorem one could hope to prove working *just* from the axioms ZFC.

The consistency with ZFC that every projective set is Lebesgue measurable, while *true*, cannot be proved assuming just the consistency of ZFC. Nevertheless, an immediate corollary of Solovay's results on the measure problem for the projective sets is that if ZFC is consistent, then so is ZFC, together with the statement

There is no paradoxical partition of the unit ball of  $\mathbb{R}^3$  into pieces, each of which is a projective set.

Thus already at  $H(\omega_1)$  there are natural structural questions which are formally unsolvable. The resolution of these questions, if indeed they can be resolved, requires the discovery of new axioms.

Both Gödel's method of restriction and Cohen's method of extension can alter  $H(\omega_1)$  in the sense of the models, even in the case that the initial and final models are transitive. Informally, restricting generally deletes sets of integers, and new sets of integers can appear in a Cohen extension. Thus it is not at all obvious that these questions about the projective sets are any more tractable than the Continuum Hypothesis.

[Gödel, 1947] There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole discipline, and furnishing such powerful methods for solving given problems (and even solving them, as far as possible, in a constructivistic way) that quite irrespective of their intrinsic necessity they would have to be assumed at least in the same sense as any established physical theory.

I now discuss one candidate for such an axiom.

### Determinacy

Fix  $A \subseteq [0, 1]$ . I define the game  $G_A$ , which is an infinite game with two players.

Player I and Player II alternate choosing

$$\epsilon_i \in \{0, 1\},$$

so that Player I chooses  $\epsilon_i$  for  $i$  odd, and Player II chooses  $\epsilon_i$  for  $i$  even. Player I begins by choosing  $\epsilon_1$ ; Player II then picks  $\epsilon_2$  and so forth.

Player I wins if

$$\sum_{i=1}^{\infty} \epsilon_i 2^{-i} \in A;$$

otherwise Player II wins.

Let SEQ be the set of all finite binary sequences. A strategy  $\tau$  is a function

$$\tau : \text{SEQ} \rightarrow \{0, 1\}.$$

A run  $\langle \epsilon_i : i \in \mathbb{N} \rangle$  of the game is generated according to  $\tau$  by Player I if  $\epsilon_1 = \tau(\emptyset)$  and for all  $k \in \mathbb{N}$ ,

$$\epsilon_{2k+1} = \tau(\epsilon_1, \dots, \epsilon_{2k}).$$

Similarly the run is generated according to  $\tau$  by Player II if for all  $k \in \mathbb{N}$ ,  $\epsilon_{2k} = \tau(\epsilon_1, \dots, \epsilon_{2k-1})$ .

A strategy  $\tau$  is a *winning strategy* for Player I if every run of the game generated according to  $\tau$  by Player I is winning for Player I. Similarly,  $\tau$  is a *winning strategy* for Player II if every run of the game generated according to  $\tau$  by Player II is winning for Player II.

**Definition.** Suppose that  $A \subseteq [0, 1]$ . The game  $G_A$  is *determined*—briefly,  $A$  is determined—if there is a winning strategy for one of the players.  $\square$

The Axiom of Choice yields a set  $A$  such that  $G_A$  is *not* determined. This is a simple diagonalization argument. There are only  $2^{\aleph_0}$  many possible strategies, and for each strategy  $\tau$  the assertion that  $\tau$  is a winning strategy for one of the players in the game  $G_A$  effectively makes  $2^{\aleph_0}$  many predictions about membership in  $A$ .

However, the undetermined set given by the Axiom of Choice is not in general a projective set. This (essentially, Mycielski–Steinhaus, 1962) suggests the following axiom:

**Projective Determinacy:** Suppose that  $A$  is a projective subset of  $[0, 1]$ . Then the game  $G_A$  is determined.

It has to be acknowledged at this point in our discussion that the axiom Projective Determinacy is not only *not* obviously true, it is not even obviously consistent. It is, however, a fruitful axiom, but (a logician's joke) so is the axiom  $0 = 1$ .

In this article, the first of a two-part series, my main objective is to present some of the compelling evidence that the axiom Projective Determinacy is the “right axiom” for the projective sets. The evidence I present is really a small fraction of what is now available. The subject of the projective sets has expanded in ways not foreseen or even imagined by its founders.

In 1964 Mycielski and Swierczkowski proved that if the axiom Projective Determinacy holds, then every projective set is Lebesgue measurable. Consequently, this axiom implies that there is no paradoxical partition of the unit ball of  $\mathbb{R}^3$  into projective pieces.

The axiom also implies that every uncountable projective set has cardinality  $2^{\aleph_0}$ . In fact much more is true. Davis proved, again shortly after the axiom was introduced, that the axiom implies

that every uncountable projective set contains an uncountable closed subset.

This establishes that there is no *formal* counterexample to CH to be found among the projective sets (assuming Projective Determinacy).

Surprisingly, the actual relationship between CH and the projective sets is quite subtle, *even* assuming Projective Determinacy. This will be the starting point for the second part of this article.

One can also naturally analyze versions of the Axiom of Choice for the projective sets.

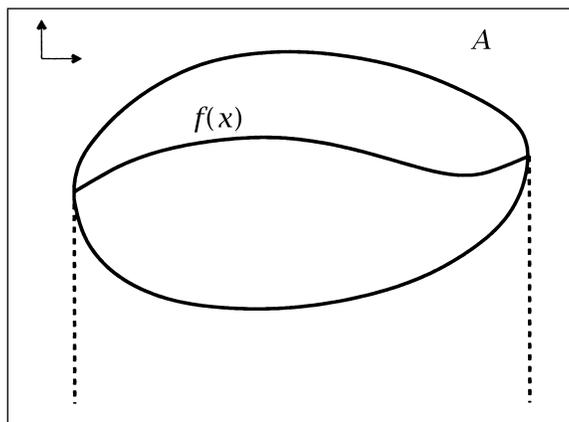
Suppose that  $A \subseteq \mathbb{R}^2$ . For each  $x \in \mathbb{R}$  let

$$A_x = \{y \in \mathbb{R} \mid (x, y) \in A\}$$

be the section of  $A$  at  $x$ .

Let  $B$  be the projection of  $A$ ,  $B = \{x \mid A_x \neq \emptyset\}$ .

A function  $f : B \rightarrow \mathbb{R}$  *uniformizes* the set  $A$  if for each  $x \in B$ ,  $f(x) \in A_x$ ; see the figure below. The



function  $f$  is projective if its graph is a projective subset of  $\mathbb{R}^2$ .

In 1930 Luzin asked if every projective subset of the plane can be uniformized by a projective function. Nearly half a century later Moschovakis proved that the answer is affirmative if Projective Determinacy holds.

Thus, assuming Projective Determinacy, one has a complete analysis of the Axiom of Choice at the projective level, which can be summarized as follows. For this summary it is convenient to generalize the notion of a projective subset of  $\mathbb{R}^n$  to the notion of a general projective set. So let's say that a set  $A$  is a *general projective set* if there exists a surjection  $\pi : \mathbb{R} \rightarrow M$ , where  $M$  is the transitive closure of  $\{A\}$ , such that

$$\{(x, y) \mid \pi(x) \in \pi(y)\}$$

is a projective subset of  $\mathbb{R}^2$ . The two notions coincide for subsets of  $\mathbb{R}^n$ ; a set  $A \subseteq \mathbb{R}^n$  is a general projective set if and only if it is a projective set.

Projective sets are by definition subsets of  $\mathbb{R}^n$ ; general projective sets can be ordinals, functions, etc. Here is the promised summary.

1. The *Axiom of Choice* fails projectively in that there is no projective well-ordering of the reals.
2. The *Axiom of Choice* holds projectively in that if  $F : \mathbb{R} \rightarrow V$  is a general projective set, then there is a choice function for  $F$  which is also a general projective set.

### Projective Determinacy and Large Cardinals

One might attempt to analyze Projective Determinacy incrementally.

In 1953 Gale-Stewart proved that every open subset of  $[0, 1]$  is determined, and they boldly asked whether every Borel set is determined. Two decades later, in a technical tour de force, Martin proved the answer is affirmative. A remarkable aspect of Martin's proof is that Friedman [Friedman, 1971] had previously shown that the determinacy of all Borel sets could *not* be proved in Zermelo set theory with the Axiom of Choice. This is the axiom system ZC, i.e., ZFC without the Axiom(s) of Replacement. Most mathematicians, realizing it or not, work in this restricted axiom system.

Roughly, Martin's method was to associate to a Borel set  $A$ , by induction on the Borel rank of  $A$ , an open set  $A^* \subset Z^{\mathbb{N}}$  where  $Z$  is discrete.  $A^*$  is constructed so that from the determinacy of the game associated to  $A^*$  (here the players play elements of  $Z$ ), one infers the determinacy of  $A$ . The Gale-Stewart Theorem applies to show that  $A^*$  is determined, and so one obtains the determinacy of  $A$ . As  $A$  ranges over the possible Borel sets, the associated sets  $Z$  range over

$$\{\mathcal{P}^\alpha(\mathbb{R}) \mid \alpha < \omega_1\},$$

increasing in cardinality with the Borel rank of  $A$ . Here  $\mathcal{P}^\alpha(\mathbb{R})$  denotes the  $\alpha$ -th iterated powerset of  $\mathbb{R}$ , which is defined by induction as follows, where for each set  $X$ ,  $\mathcal{P}(X) = \{Y \mid Y \subseteq X\}$ .  $\mathcal{P}(X)$  is the powerset of  $X$ .  $\mathcal{P}^0(\mathbb{R}) = \mathbb{R}$ ,  $\mathcal{P}^{\alpha+1}(\mathbb{R}) = \mathcal{P}(\mathcal{P}^\alpha(\mathbb{R}))$ , and

$$\mathcal{P}^\alpha(\mathbb{R}) = \cup \{\mathcal{P}^\beta(\mathbb{R}) \mid \beta < \alpha\}$$

if  $\alpha > 0$  and  $\alpha$  is a limit ordinal.

In Zermelo set theory one cannot prove that  $\mathcal{P}^\omega(\mathbb{R})$  even exists, and so as Friedman's theorem predicted, Martin's proof cannot be implemented assuming just the axioms ZC.

The determinacy of all analytic sets  $A \subseteq [0, 1]$  cannot be proved in ZFC. This is because the theory ZFC is simply not strong enough. Thus Martin's theorem on the determinacy of all Borel sets is the strongest theorem one can hope to prove without appealing to new axioms. Here enter large cardinal axioms, which informally are axioms which assert the existence of "large" cardinals. Perhaps best known among these is the axiom which asserts the existence of a measurable cardinal. An uncountable cardinal  $\kappa$  is a *measurable cardinal* if there exists a nonprincipal ultrafilter  $U$  on the subsets of  $\kappa$  which is  $\kappa$ -complete; i.e., if

$X \subset U$  and  $X$  has cardinality less than  $\kappa$ , then  $\cap\{A \subset \kappa \mid A \in X\} \in U$ .

Around 1970, almost five years before he proved all Borel sets are determined, Martin proved that if there is a measurable cardinal, then all analytic sets are determined (every Borel set is an analytic set, and so this also established that all Borel sets are determined, but *not* in the theory ZFC).

But Solovay showed that one cannot hope to prove Projective Determinacy from just the existence of a measurable cardinal. The reason, as before, is a lack of strength: ZFC + “Projective Determinacy” implies the consistency of ZFC + “There is a measurable cardinal”. Therefore, by Gödel’s Second Incompleteness Theorem one cannot infer Projective Determinacy from the existence of a measurable cardinal. Still stronger axioms are necessary, and the special case of proving the determinacy of all  $\Sigma_2^1$  subsets of  $[0, 1]$  became a central problem (the  $\Sigma_2^1$  sets are those projective sets which can be represented as the projection of the complement of an analytic set). Some speculated that *no* large cardinal axiom *known* was sufficient in strength to imply this fragment of Projective Determinacy. In 1978 Martin succeeded in proving the determinacy of all  $\Sigma_2^1$  sets by using essentially the strongest large cardinal hypothesis known at the time. Finally, in 1983 I proved the determinacy of all projective sets using large cardinal axioms in a natural hierarchy which begins with the large cardinal axiom used by Martin to establish the determinacy of all  $\Sigma_2^1$  sets.

The natural character of these determinacy proofs suggested that essentially optimal large cardinal assumptions were being used. But this picture, though quite appealing (at least to Martin and me), was completely wrong. The large cardinal axioms used to prove Projective Determinacy, including the axiom used by Martin to prove the determinacy of all  $\Sigma_2^1$  sets, were *far stronger* than necessary. The first indications that the picture was incorrect came from a surprising direction: the evidence was discovered by Foreman–Magidor–Shelah in their seminal work on Martin’s Maximum, a maximal forcing axiom which I shall discuss below. This ultimately led to the following theorem from 1984.

**Theorem (Shelah–Woodin).** *Assume there exist infinitely many Woodin cardinals. Then every projective set is Lebesgue measurable.*  $\square$

I shall not give the definition of a Woodin cardinal, but instead simply note that this large cardinal axiom is much weaker than those used in the determinacy proofs just discussed.

The *Inner Model Program* attempts to generalize Gödel’s substructure construction to models satisfying various large cardinal axioms (and the stronger the axiom, the harder the problem). More

precisely, fixing a large cardinal axiom  $\Psi$ , for each model  $\langle M, E \rangle$  of ZFC +  $\Psi$  one seeks a substructure  $\langle M^*, E \cap (M^* \times M^*) \rangle$  which is also a model of ZFC +  $\Psi$  and which satisfies various sentences true in Gödel’s substructure. For example, if  $\Psi$  is the large cardinal axiom “There is a Woodin cardinal”, simply requiring that in the substructure produced, the sentence “There is a projective set which is not Lebesgue measurable” holds yields a difficult problem. For stronger large cardinal axioms there are generalizations of this requirement which yield similarly nontrivial problems.

For the large cardinal axiom “There is a measurable cardinal”, the correct generalization of Gödel’s construction was discovered by Solovay and then further analyzed in work of Kunen and Silver. With these results the Inner Model Program began.

The fact that from infinitely many Woodin cardinals one can prove the projective sets are Lebesgue measurable is strong evidence that from the same assumption one should be able to prove Projective Determinacy. In 1985, using techniques developed in the Inner Model Program, Martin–Steel succeeded in doing this. Surprisingly, the combinatorial properties of Woodin cardinals responsible for their discovery—for example, those aspects yielding the measurability of all projective sets—play no role in this determinacy proof.

**Theorem (Martin–Steel).** *Assume there exist infinitely many Woodin cardinals. Then every projective set is determined.*  $\square$

How are large cardinals used in determinacy proofs? The basic strategy is the same as for Martin’s proof of the determinacy of all Borel sets, though Martin’s proof of the determinacy of all analytic sets from the existence of a measurable cardinal is a more accurate prototype. Given a set  $A \subseteq [0, 1]$ , one again associates to the set  $A$  an open set  $A^* \subset Z^{\mathbb{N}}$ , where  $Z$  is a discrete set carefully constructed such that from the determinacy of the game naturally associated to  $A^*$ , one obtains the determinacy of the initial set  $A$ . The Gale–Stewart Theorem shows that  $A^*$  is determined, and so as before one obtains the determinacy of  $A$ . For a typical projective set  $A$  the associated set  $Z$  is *very* large, of cardinality on the order of the large cardinals one is assuming to exist.

The connection between Projective Determinacy and large cardinal axioms is fundamental. This claim is supported by the following theorem from 1987, which shows that this time the picture is correct.

**Theorem (Woodin).** *The following are equivalent:*

1. Projective Determinacy.
2. For each  $k \in \mathbb{N}$  there exists a countable transitive set  $M$  such that

$\langle M, \varepsilon \rangle \models \text{ZFC} + \text{“There exist } k \text{ Woodin cardinals”}$

and such that  $M$  is countably iterable.  $\square$

The notion that  $M$  is *countably iterable* is a technical one from the Inner Model Program.

The *Core Model Program*, which is more ambitious than the Inner Model Program, originates in seminal work of Dodd and Jensen. It is more ambitious simply because it attempts to build the substructures of the Inner Model Program *without* necessarily assuming that the (relevant) large cardinal axioms even hold in the initial structure.

The extension of the Core Model Program to the realm of Woodin cardinals is due primarily to Steel [Steel, 1996], and with this development it has become clear that Projective Determinacy is actually implied by a vast number of seemingly unrelated combinatorial propositions. Projective Determinacy is ubiquitous in set theory.

In fact, this is an instance of a more pervasive phenomenon where propositions are calibrated, on the basis of consistency, by large cardinal axioms. There are now numerous examples of this phenomenon. Many early results of this kind were proved using Jensen’s *Covering Lemma*. In fact, using the *Covering Lemma*, one can prove the determinacy of all analytic sets from a perhaps bewildering array of propositions. I refer the reader to [Kanamori, 1994] for further details.

A recent example where methods from the Core Model Program are used to establish Projective Determinacy involves axioms which attempt to solve the continuum problem by explicitly making the Continuum Hypothesis false.

### Forcing Axioms

*Forcing Axioms* are in essence axioms asserting generalizations of the *Baire Category Theorem*. The connection lies in the technical aspects of Cohen’s method of constructing extensions of models of ZFC.

Suppose that  $\langle M, E \rangle \models \text{ZFC}$ . The Cohen extensions associated to the structure  $\langle M, E \rangle$  correspond to complete Boolean algebras in the sense of  $\langle M, E \rangle$ , i.e., to elements  $b \in M$  such that

$$\langle M, E \rangle \models \text{“}b \text{ is a complete Boolean algebra”}.$$

If  $b$  is trivial, for example, if

$$\langle M, E \rangle \models \text{“}b \text{ is a finite Boolean algebra”},$$

then the associated extension is simply  $\langle M, E \rangle$ . But if

$$\langle M, E \rangle \models \text{“}b \text{ is the measure algebra}^1 \text{ of the product space } \prod_{\omega_2} [0, 1]\text{”},$$

<sup>1</sup>The Boolean algebra of Borel subsets modulo null sets.

then the Continuum Hypothesis necessarily *fails* in the associated extension. Another quite interesting feature of this extension is that in this extension there exists a countably additive measure on  $\mathbb{R}^3$ , *extending* Lebesgue measure, which measures all projective sets and which is invariant under rigid motions. So in this extension the following statement holds:

There is no paradoxical partition of the unit ball of  $\mathbb{R}^3$  into projective pieces.

This extension, first defined and analyzed by Solovay, is sometimes referred to as a *Solovay extension*.

If  $\Omega$  is a compact Hausdorff space, then the *regular open algebra* of  $\Omega$  is the complete Boolean algebra given by the lattice of regular open subsets of  $\Omega$  (open sets  $O \subseteq \Omega$  which are the interior of their closure) ordered by set inclusion. Forcing axioms are naturally motivated by the following feature of Cohen’s original extension; this extension is defined from  $b \in M$  such that

$$\langle M, E \rangle \models \text{“}b \text{ is the regular open algebra of the product space } \prod_{\omega_2} [0, 1]\text{”}.$$

The interesting feature of this extension is that in it the following generalization of the Baire Category Theorem holds:

The unit interval  $[0, 1]$  is not the  $\aleph_1$  union of meager sets.

A compact Hausdorff space  $\Omega$  is *ccc* if every collection of pairwise disjoint open subsets of  $\Omega$  is countable.

**Martin’s Axiom** ( $\omega_1$ ): Suppose that  $\Omega$  is a compact Hausdorff space which is *ccc*. Then  $\Omega$  is not the union of  $\aleph_1$  many meager subsets of  $\Omega$ .

A simple motivation for such an axiom is that if CH is to be false, then sets of cardinality  $\aleph_1$  should behave, as much as possible, like countable sets.

One can attempt to strengthen the axiom, allowing a larger class of compact Hausdorff spaces. But one *cannot* allow arbitrary compact Hausdorff spaces. The maximum possible class was identified by Foreman–Magidor–Shelah. The definition involves the notion of a closed unbounded subset of  $\omega_1$ : a cofinal set  $C \subset \omega_1$  is *closed and unbounded* if it is closed in the order topology of  $\omega_1$ .

Suppose that  $\mathbb{B}$  is a complete Boolean algebra. Every set  $S \subseteq \mathbb{B}$  has a greatest lower bound, denoted by  $\bigwedge S$ , and a least upper bound which is denoted by  $\bigvee S$ .

**Definition (Foreman–Magidor–Shelah).** Suppose that  $\mathbb{B}$  is a complete Boolean algebra. The Boolean algebra  $\mathbb{B}$  is *stationary set preserving* if the following holds. Suppose that  $b$  is a nonzero element of  $\mathbb{B}$  and that  $\langle b_\alpha : \alpha < \omega_1 \rangle$  is a sequence of

elements of  $\mathbb{B}$ . Then there exist  $0 < c \leq b$  such that either  $c \wedge b_\alpha = 0$  for sufficiently large  $\alpha$  or there exists a closed unbounded set  $C \subset \omega_1$  such that for all  $y \in C$

$$c \wedge \left( \bigwedge_{\alpha < y} \left( \bigvee_{\alpha < \eta < y} b_\eta \right) \right) \neq 0. \quad \square$$

The class of compact Hausdorff spaces whose regular open algebras are stationary set preserving is the largest class for which one can hope to generalize Martin's Axiom( $\omega_1$ ).

**Theorem (Foreman–Magidor–Shelah).** *Suppose that  $\Omega$  is a compact Hausdorff space and that for each open, nonempty set  $O \subseteq \Omega$  the set  $O$  is not the union of  $\aleph_1$  many meager subsets of  $\Omega$ . Then the regular open algebra of  $\Omega$  is stationary set preserving.*  $\square$

This suggests the definition of Martin's Maximum. The main theorem of [Foreman–Magidor–Shelah, 1988] is that the consistency of this axiom with ZFC follows from the consistency, with ZFC, of (suitable) large cardinal axioms; in other words, this maximum can be realized.

**Martin's Maximum:** Suppose that  $\Omega$  is a compact Hausdorff space whose regular open algebra is stationary set preserving. Then  $\Omega$  is not the union of  $\aleph_1$  many meager subsets of  $\Omega$ .

Forcing Axioms by design imply that the Continuum Hypothesis is false. In fact, Martin's Maximum (as opposed to the weaker Martin's Axiom( $\omega_1$ )) gives quite a bit more information about the cardinality of the continuum.

**Theorem (Foreman–Magidor–Shelah).** *Suppose that the axiom Martin's Maximum holds. Then  $2^{\aleph_0} = \aleph_2$ .*  $\square$

Subsequent investigations have revealed that this theorem actually holds for almost any forcing axiom which is nontrivially stronger than Martin's Axiom( $\omega_1$ ). So, curiously, insisting that sets of cardinality  $\aleph_1$  resemble countable sets necessitates that  $2^{\aleph_0} = \aleph_2$ .

The large cardinal axioms used to establish the consistency of Martin's Maximum are far stronger than the large cardinal axioms used to prove Projective Determinacy. It is therefore natural to speculate that Martin's Maximum might *imply* Projective Determinacy, even though Martin's Maximum is *not* a large cardinal axiom in the accepted sense of what a large cardinal axiom is.

I noted that the work of Foreman–Magidor–Shelah on Martin's Maximum inspired the discovery of the correct large cardinal axioms for Projective Determinacy. The next theorem is therefore perhaps a fitting conclusion to this part of the story. I state a slightly stronger version involving a weakening of Martin's Maximum. The weakening is Martin's

Maximum( $c$ ), which is the axiom that Martin's Maximum holds for all compact Hausdorff spaces  $\Omega$  whose regular open algebra is stationary set preserving and for which there is a base for the topology of  $\Omega$  with cardinality at most  $c$ . Thus Martin's Maximum( $c$ ) is an axiom concerned with only relatively “small” compact Hausdorff spaces.

**Theorem (Woodin).** *Suppose that the axiom Martin's Maximum( $c$ ) holds. Then Projective Determinacy holds.*  $\square$

There is no known direct proof of this theorem. The method of the proof is to use machinery developed in the Core Model Program to show, assuming Martin's Maximum( $c$ ), that for each  $n < \omega$  there exists a countable transitive set  $M$  such that

$\langle M, \in \rangle \models \text{ZFC} + \text{“There exist } n \text{ Woodin cardinals”}$

and such that  $M$  is (countably) iterable.

One obtains Projective Determinacy from the existence of these sets by, essentially, the theorem unifying Projective Determinacy with Large Cardinals. This methodology can and has been used to prove Projective Determinacy from a large number of combinatorial statements, many of which, like Martin's Maximum, have no obvious relationship to the projective sets. Hence the claim: Projective Determinacy is ubiquitous in set theory.

To summarize the current state of affairs for the theory of the projective sets:

- Projective Determinacy is the **correct** axiom for the projective sets; the ZFC axioms are *obviously* incomplete and, moreover, incomplete in a fundamental way.
- Assuming Projective Determinacy, there are no essential uses of the Axiom of Choice in the analysis of the structure  $\langle H(\omega_1), \in \rangle$  (or, equivalently, in the analysis of  $\langle \mathcal{P}(\mathbb{N}), \mathbb{N}, +, \cdot, \in \rangle$ , the standard structure for Second Order Number Theory).
- The only known examples of unsolvable problems about the projective sets, in the context of Projective Determinacy, are analogous to the known examples of unsolvable problems in number theory: Gödel sentences and consistency statements.

In brief, one has the following analogy of axioms for structures:<sup>2</sup>

$$\frac{\text{Peano Axioms}}{\langle \mathbb{N}, +, \cdot \rangle} \sim \frac{\text{Projective Determinacy} + \text{Peano Axioms}}{\langle \mathcal{P}(\mathbb{N}), \mathbb{N}, +, \cdot, \in \rangle}.$$

Can the understanding of the structure  $\langle H(\omega_1), \in \rangle$ , achieved through the discovery of the

<sup>2</sup>Here the precise formulation of Projective Determinacy incorporates  $\Sigma_0$ -Comprehension.

correct axioms for this structure, be extended to an understanding of the structure  $\langle H(\omega_2), \in \rangle$ ? This question will be the main topic of the second, and final, part of this article.

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