EQUIVALENCE OF NORMS

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1. Introduction

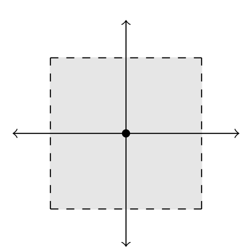
Let $(K, |\cdot|)$ be a valued field and V be a K-vector space. A vector space norm on V is a function $\|\cdot\| : V \times V \to \mathbf{R}$ such that

(1) $\|v\| \geq 0$ for all $v \in V$, with equality if and only if $v = \mathbf{0}$.
(2) $\|v + w\| \leq \|v\| + \|w\|$ for all v and w in V,
(3) $\|cv\| = |c| \, \|v\|$ for all $c \in K$ and $v \in V$.

A norm $\|\cdot\|$ on V defines a metric on V by $d(v, w) = \|v - w\|$, from which we get open subsets and closed subsets of V relative to $\|\cdot\|$. Different norms always define different metrics since the norm can be recovered from the metric it defines ($\|v\| = d(v, 0)$), but different norms might define the same concept of open subset of V.

Example 1.1. On $\mathbf{R}^2$ let $\|\cdot\|$ be the usual Euclidean norm. Another norm on $\mathbf{R}^2$ is $\|v\|' = 2\|v\|$. The condition $\|v - a\|' < r$ is the same as $\|v - a\| < r/2$, so open balls in $\mathbf{R}^2$ defined by $\|\cdot\|$ and $\|\cdot\|$ are the same (even if the radii don't match). Open subsets for a norm are unions of open balls for that norm, so the open subsets of $\mathbf{R}^2$ for $\|\cdot\|$ and $\|\cdot\|'$ coincide.

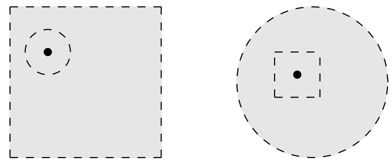
Example 1.2. On $\mathbf{R}^2$ let $\|\cdot\|$ be the usual Euclidean norm and set $\|(x, y)\|' = \max(|x|, |y|)$. It is left to the reader to check $\|\cdot\|'$ is a norm on $\mathbf{R}^2$. This is not a scalar multiple of the Euclidean norm on $\mathbf{R}^2$: an open ball centered at the origin for $\|\cdot\|'$ is an open square (no boundary) centered at the origin with sides parallel to the axes. See the figure below. More generally, open balls for $\|\cdot\|'$ are open squares in $\mathbf{R}^2$ with sides parallel to the axes.



Unlike in Example 1.1, open balls for $\|\cdot\|$ and $\|\cdot\|'$ are not the same, but each is an open subset for the other norm:

- In each open ball for $\|\cdot\|'$, which is an ordinary open square, every point is contained in a Euclidean open ball that is inside the square.
- In a Euclidean open ball, each point is contained in an open square with sides parallel to the axes that is inside the Euclidean ball.

The picture below illustrates these two scenarios.



Thus the open subsets (not open balls!) of $\mathbf{R}^2$ for $\|\cdot\|$ and $\|\cdot\|'$ coincide.

Definition 1.3. Two norms on a K-vector space V are called *equivalent* if they define the same open subsets of V.

Our goal is two-fold: (i) describe equivalence of norms by a criterion analogous to the formula $|\cdot|' = |\cdot|^t$ for $t > 0$ linking equivalent absolute values $|\cdot|$ and $|\cdot|'$ on a field, and (ii) show all norms on a finite-dimensional vector space over a complete valued field are equivalent. The analogue of $|\cdot|' = |\cdot|^t$ will not be a sharp equation, but a pair of inequalities.

2. Equivalent norms

Theorem 2.1. *Let $(K, |\cdot|)$ be a nontrivially valued field and V be a K-vector space. Two norms $\|\cdot\|$ and $\|\cdot\|'$ on V are equivalent if and only if there are constants $A > 0$ and $B > 0$ such that $A\|v\|' \le \|v\| \le B\|v\|'$ for all $v \in V$.*

We can rewrite the inequalities in other ways:

- $(1/B)\|v\| \le \|v\|' \le (1/A)\|v\|$ for all $v \in V$ (this exchanges $\|\cdot\|$ and $\|\cdot\|'$)
- $\|v\|' \le C\|v\|$ (for $C = 1/A$) and $\|v\| \le C'\|v\|'$ (for $C' = B$) for all $v \in V$.

Proof. The theorem is obvious if $V = \{0\}$, so assume V is not $\{0\}$.

($\Longleftarrow$) First assume there are positive A and B such that $A\|v\|' \le \|v\| \le B\|v\|'$ for all $v \in V$. Then for each open set $U \subset V$ with respect to $\|\cdot\|$ and $v \in U$, there is an $\varepsilon > 0$ such that the open ε-ball with respect to $\|\cdot\|$ around v is contained in U:

$$\{w \in V : \|w - v\| < \varepsilon\} \subset U.$$

Since

$$\|w - v\|' < \frac{\varepsilon}{B} \Longrightarrow \|w - v\| < \varepsilon,$$

each open $\|\cdot\|$-ball around v contains an open $\|\cdot\|'$-ball around v, so U is open with respect to $\|\cdot\|'$. The proof that each open subset of V with respect to $\|\cdot\|'$ is open with respect to $\|\cdot\|$ is similar, using $\|v\|' \le (1/A)\|v\|$ in place of $\|v\| \le B\|v\|'$.

($\Longrightarrow$) Assume $\|\cdot\|$ and $\|\cdot\|'$ are equivalent. Then the open unit ball around the origin in V relative to $\|\cdot\|$ is open relative to $\|\cdot\|'$ and the open unit ball around the origin in V relative to $\|\cdot\|'$ is open relative to $\|\cdot\|$, so there are $r > 0$ and $s > 0$ such that

$$\{v \in V : \|v\|' < r\} \subset \{v \in V : \|v\| < 1\}, \quad \{v \in V : \|v\| < s\} \subset \{v \in V : \|v\|' < 1\}. \tag{2.1}$$

Since $|\cdot|$ is nontrivial on K, there is $\gamma \in K$ such that $|\gamma| > 1$. Then $|\gamma|^n \to \infty$ as $n \to \infty$ and $|\gamma|^n \to 0$ as $n \to -\infty$, so every positive real number is between two successive powers of $|\gamma|$ or equal to one of these powers. Therefore for each nonzero $v \in V$ there is an integer n such that

$$|\gamma|^n \le \frac{1}{s}\|v\| < |\gamma|^{n+1}. \tag{2.2}$$

Then
$$\left\|\frac{1}{\gamma^{n+1}}v\right\| = \frac{1}{|\gamma|^{n+1}}\|v\| < s,$$
so $\|v/\gamma^{n+1}\|' < 1$ by (2.1). Thus $\|v\|' < |\gamma|^{n+1} = |\gamma||\gamma|^n \leq (|\gamma|/s)\|v\|$. Setting $B = |\gamma|/s$ we have $\|v\|' < B\|v\|$ for all nonzero $v \in V$, so $\|v\|' \leq B\|v\|$ for all v, including $v = 0$.

In a similar way, switching the roles of $\|\cdot\|$ and $\|\cdot\|'$ and using r in place of s, we get $\|v\| \leq (|\gamma|/r)\|v\|'$ for all $v \in V$, so $A\|v\| \leq \|v\|'$ where $A = r/|\gamma|$. □

Theorem 2.1 is about norms on a vector space over a nontrivially valued field. Could the theorem have a counterexample when K is equipped with the trivial absolute value? Not if $\dim_K(V)$ is finite (see Theorem 3.2, which is true for trivially valued fields since they are complete), but here is a counterexample for an infinite-dimensional vector space.

Example 2.2. Let K be a field with the trivial absolute value and $V = \{(c_0, c_1, c_2, \ldots) : c_i \in K\}$ be the K-vector space of all sequences in K. For $0 < \varepsilon < 1$, define the ε-norm on V as follows:
$$\|(c_0, c_1, c_2, \ldots)\|_\varepsilon = \begin{cases} 0, & \text{if all } c_i = 0, \\ \varepsilon^n, & \text{if } c_n \neq 0, c_i = 0 \text{ for } i < n. \end{cases}$$
It is left to the reader to check this is a K-vector space norm on V.[1] (That K is trivially valued is relevant because by definition $\|cv\|_\varepsilon = \|v\|_\varepsilon$ for all $c \in K^\times$ and $v \in V$, and $|c|\|v\|_\varepsilon = \|v\|_\varepsilon$ when $c \in K^\times$ and K is trivially valued.)

The topology put on V by $\|\cdot\|_\varepsilon$ does not depend on ε: if ε and δ are both in $(0, 1)$, then $\delta = \varepsilon^t$ for some $t > 0$ and a short calculation shows $\|v\|_\delta = \|v\|_\varepsilon^t$ for all $v \in V$, so $\|v\|_\delta < r \Leftrightarrow \|v\|_\varepsilon < r^{1/t}$. Thus open balls around $\mathbf{0}$ in one norm are open balls around $\mathbf{0}$ in the other norm, and by translation the same is true for open balls around each point in V.

For distinct ε and δ in $(0, 1)$, the norms $\|\cdot\|_\varepsilon$ and $\|\cdot\|_\delta$ define the same topology on V, so they are equivalent norms in the sense of Definition 1.3, but it is *not* true that there are $A, B > 0$ such that $A\|v\|_\varepsilon \leq \|v\|_\delta \leq B\|v\|_\varepsilon$ for all $v \in V$. Indeed, suppose there were such A and B. For $n \geq 0$ let $v_n = (0, 0, \ldots, 0, 1, 0, \ldots)$ have 1 in component n and 0 in all other components, so when $v = v_n$ the inequalities $A\|v\|_\varepsilon \leq \|v\|_\delta \leq B\|v\|_\varepsilon$ imply $A\varepsilon^n \leq \delta^n \leq B\varepsilon^n$. Therefore $A \leq (\delta/\varepsilon)^n \leq B$, and this holds for all $n \geq 0$. Thus the powers $(\delta/\varepsilon)^n$ are bounded away from both 0 and ∞, which is false: if $\delta < \varepsilon$ then $(\delta/\varepsilon)^n \to 0$ as $n \to \infty$, while if $\delta > \varepsilon$ then $(\delta/\varepsilon)^n \to \infty$ as $n \to \infty$.

3. Norms on finite-dimensional spaces

We want to look at general norms on finite-dimensional spaces over complete fields, but first we will look at a particular construction of a norm using a choice of basis.

Lemma 3.1. *Let V be a finite-dimensional vector space over a valued field $(K, |\cdot|)$. For a basis $\{e_1, \ldots, e_n\}$ of V over K, set*
$$\left\|\sum_{i=1}^n c_i e_i\right\|_\infty = \max_{1\leq i\leq n} |c_i| \tag{3.1}$$
where $c_i \in K$. This is a norm on V, and if K is complete with respect to $|\cdot|$ then V is complete with respect to $\|\cdot\|_\infty$.

[1]If we identify the formal power series ring $K[[X]]$ with V by $\sum_{n\geq 0} c_n X^n \leftrightarrow (c_0, c_1, c_2, \ldots)$, then the ε-norm on V corresponds to the X-adic norm on $K[[X]]$ where $|X| = \varepsilon$.

Proof. The axioms of a vector space norm are easy to verify for $\|\cdot\|_\infty$, *e.g.*, the triangle inequality for $\|\cdot\|_\infty$ follows from the triangle inequality for $|\cdot|$ on basis coefficients.

Suppose K is complete with respect to $|\cdot|$. To prove V is complete with respect to $\|\cdot\|_\infty$, let $\{v_m\}$ be a Cauchy sequence in V with respect to $\|\cdot\|_\infty$: for each $\varepsilon > 0$ there is an $M \geq 1$ such that $\ell, m \geq M \Longrightarrow \|v_\ell - v_m\|_\infty < \varepsilon$. Writing v_m in terms of the basis as

$$v_m = c_{1m}e_1 + c_{2m}e_2 + \cdots + c_{nm}e_n$$

the definition of the norm $\|\cdot\|_\infty$ tells us

$$\ell, m \geq M \Longrightarrow |c_{i\ell} - c_{im}| < \varepsilon$$

for each i from 1 to n, so each sequence $\{c_{im}\}_{m\geq 1}$ in K is Cauchy with respect to $|\cdot|$. Since K is complete with respect to $|\cdot|$, there is a limit: $\lim_{m\to\infty} c_{im} = c_i$ in $(K, |\cdot|)$. Then $v = \sum c_i e_i$ is a limit for the v_m's in V:

$$\|v - v_m\|_\infty = \max_{1\leq i\leq n} |c_i - c_{im}| \to 0 \text{ as } m \to \infty.$$

□

Here is the most important property of norms on finite-dimensional spaces.

Theorem 3.2. *All norms on a finite-dimensional vector space over a complete valued field are equivalent.*

Proof. Let $(K, |\cdot|)$ be a complete valued field and V be a K-vector space. To prove all norms on V are equivalent, we use induction on $\dim_K V$. The case $V = \{0\}$ is trivial, so we can assume $\dim_K V \geq 1$. If $\dim_K V = 1$ and v_0 is nonzero in V then $V = Kv_0$ and $\|cv_0\| = |c| \|v_0\|$ for $c \in K$. So two norms $\|\cdot\|$ and $\|\cdot\|'$ on V must be scalar multiples of each other: $\|v\|' = (\|v_0\|' / \|v_0\|) \|v\|$ for all $v \in V$.

Assume $n \geq 2$ and the theorem is proved for all K-vector spaces of dimension $n - 1$. Let V have dimension n. Pick a K-basis $\{e_1, \ldots, e_n\}$ and define a norm $\|\cdot\|_\infty$ from this basis by (3.1). For an arbitrary norm $\|\cdot\|$ on V, we will show there are constants $A > 0$ and $B > 0$ such that $A\|v\|_\infty \leq \|v\| \leq B\|v\|_\infty$ for all $v \in V$; thus every norm on V is equivalent to $\|\cdot\|_\infty$, so all norms on V are equivalent to each other. The existence of B will be very easy, but the existence of A will be proved indirectly by contradiction.

For each $v \in V$ write $v = c_1e_1 + \cdots + c_ne_n$ with $c_i \in K$. Then

$$\|v\| \leq \sum_{i=1}^{n} \|c_ie_i\| = \sum_{i=1}^{n} |c_i| \|e_i\| \leq B \max |c_i|,$$

where $B = \sum \|e_i\| > 0$, so $\|v\| \leq B\|v\|_\infty$ for all $v \in V$.

To show there is an $A > 0$ such that $A\|v\|_\infty \leq \|v\|$ for all $v \in V$, assume no $A > 0$ fits that inequality for all v. Taking $A = 1/k$ for $k = 1, 2, 3, \ldots$, that none of these can work would mean for each $k \geq 1$ there is $v_k \in V$ such that $\|v_k\| < (1/k)\|v_k\|_\infty$. We are going to pass to a subsequence and normalize these vectors in order to reach a contradiction.

By the definition of $\|\cdot\|_\infty$ in terms of the basis $\{e_1, \ldots, e_n\}$, each $\|v_k\|_\infty$ is the absolute value of some e_i-coefficient of v_k. Since the list $\{v_k\}$ is infinite (even if there are repetitions) and there are finitely many e_i, there is an index i from 1 to n such that infinitely many v_k's have $\|v_k\|_\infty$ equal to its e_i-coefficient. Without loss of generality, $i = n$. Scaling v_k by an element of $K^\times$ does not affect the inequality $\|v_k\| < (1/k)\|v_k\|_\infty$ since both sides change in the same way, so after passing to a subsequence $\{v_{k_j}\}$ where $\|v_{k_j}\|_\infty$ is the absolute value of the e_n-coefficient of v_{k_j} and then dividing v_{k_j} by its e_n-coefficient, we have a sequence v'_{k_j} in V for $j = 1, 2, 3, \ldots$ such that

(1) $\left\|v'_{k_j}\right\|_\infty = 1$,
(2) the e_n-coefficient of v'_{k_j} is 1,
(3) $\left\|v'_{k_j}\right\| < (1/k_j)\left\|v'_{k_j}\right\|_\infty = 1/k_j$.

Since $k_j \to \infty$ as $j \to \infty$, by (3) we have $\left\|v'_{k_j}\right\| \to 0$ as $j \to \infty$. Set $w_j = v'_{k_j} - e_n$, so w_j belongs to the subspace $W := \sum_{i=1}^{n-1} Ke_i$ and (3) tells us

$$\|w_j + e_n\| = \left\|v'_{k_j}\right\| \to 0 \tag{3.2}$$

as $j \to \infty$. We can remove e_n from here by taking differences:

$$\|w_j - w_{j'}\| = \|(w_j + e_n) - (w_{j'} + e_n)\| \leq \|w_j + e_n\| + \|w_{j'} + e_n\| \to 0 \tag{3.3}$$

as $j, j' \to \infty$. Thus $\{w_j\}$ is a Cauchy sequence in W with respect to the norm $\|\cdot\|$ *restricted to* W.

Since W has dimension $n-1$, all norms on W are equivalent by induction. Therefore $\|\cdot\|$ and $\|\cdot\|_\infty$ restricted to W are equivalent norms, so $\|\cdot\|_\infty \leq C\|\cdot\|$ on W for some $C > 0$ and $\|\cdot\| \leq C'\|\cdot\|_\infty$ on W for some $C' > 0$. From the first bound we get

$$\|w_j - w_{j'}\|_\infty \leq C\|w_j - w_{j'}\| \to 0$$

as $j, j' \to \infty$ by (3.3), so $\{w_j\}$ is Cauchy in W with respect to $\|\cdot\|_\infty$ restricted to W. Since $(K, |\cdot|)$ is complete, $(W, \|\cdot\|_\infty)$ is complete by Lemma 3.1, so there is a limit w for the w_j in W with respect to $\|\cdot\|_\infty$: $\|w - w_j\|_\infty \to 0$ as $j \to \infty$. Then $\|w - w_j\| \leq C'\|w - w_j\|_\infty \to 0$, so $\|w - w_j\| \to 0$. Combining this with (3.2), as $j \to \infty$

$$\|w + e_n\| = \|(w - w_j) + (w_j + e_n)\| \leq \|w - w_j\| + \|w_j + e_n\| \to 0 + 0 = 0,$$

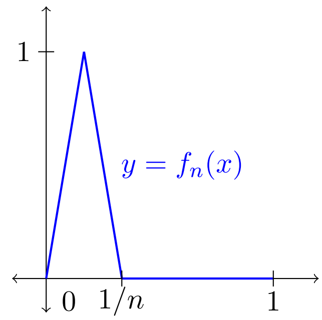
so $w = -e_n$, but $w \in W$ and $-e_n$ is not in W. We have a contradiction. $\square$

This theorem did not rely on the harder direction of Theorem 2.1: it directly shows the two norms are bounded by constant multiples of each other, and hence are equivalent by the easier direction of Theorem 2.1. In particular, the proof even works for a trivially valued field, which is always complete.

Remark 3.3. Theorem 3.2 is false for infinite-dimensional spaces. For example, let $K = \mathbf{R}$ and $V = C([0,1])$ be the vector space of continuous functions $f\colon [0,1] \to \mathbf{R}$. Define two norms on $C([0,1])$:

$$\|f\| = \max_{0 \leq x \leq 1} |f(x)|, \quad \|f\|_1 = \int_0^1 |f(x)|\,dx.$$

Clearly $\|f\|_1 \leq \|f\|$, but there is no $C > 0$ such that $\|f\| \leq C\|f\|_1$ for all f. Indeed, let f be a function that is 0 outside of $[0, 1/n]$ and have a triangular shape over $[0, 1/n]$ with a peak at height 1, as in the picture below. Then $\|f\| = 1$ and $\|f\|_1 = 1/(2n)$. No $C > 0$ can satisfy $1 \leq C/(2n)$ for all $n \geq 1$.



Corollary 3.4. *Let $(K, |\cdot|)$ be a nontrivial complete valued field and $(V, \|\cdot\|)$ be a finite-dimensional normed K-vector space. Every subspace of V is closed with respect to $\|\cdot\|$.*

Proof. Let W be a subspace of V. To say W is closed with respect to $\|\cdot\|$ is unchanged if $\|\cdot\|$ is replaced by an equivalent norm on V (this won't change the meaning of an open subset of V, so by passing to complements it won't change the meaning of a closed subset of V). By Theorem 3.2, all norms on V are equivalent, so it suffices to prove the theorem using a norm on V that is adapted to the choice of W.

Let $n = \dim_K V$ and $d = \dim_K W$. The result is obvious if $d = 0$, since then $W = \{0\}$ and a one-point subset of a normed vector space — or of a metric space – is closed (the complement is open). The result is also obvious if $d = n$, since then $W = V$ and V is a closed subset of V. We can now suppose $n \geq 2$ and $1 \leq d \leq n - 1$.

Pick a basis $\{e_1, \ldots, e_d\}$ of W and extend it to a basis $\{e_1, \ldots, e_d, e_{d+1}, \ldots, e_n\}$ of V. Let $\|\cdot\|_\infty$ be the norm on V defined from this basis by (3.1). Then $\|\cdot\|$ is equivalent to $\|\cdot\|_\infty$ by Theorem 3.2 and

$$W = \left\{ \sum_{i=1}^{n} c_i e_i : c_{d+1} = \cdots = c_n = 0 \right\}.$$

Let's show W is closed with respect to $\|\cdot\|_\infty$. If $\{w_m\} \subset W$ and $w_m \to v$ in $(V, \|\cdot\|_\infty)$, write $v = a_1 e_1 + \cdots + a_n e_n$ with $a_i \in K$. Since the e_i-coefficient of w_m is 0 for $i > d$, by the definition of $\|\cdot\|_\infty$ we have

$$|a_i| = |0 - a_i| \leq \|w_m - v\|_\infty$$

for $i = d + 1, \ldots, n$. Since $\|w_m - v\|_\infty \to 0$ as $m \to \infty$ we get $|a_i| = 0$ so $a_i = 0$ when $i \geq d + 1$. Thus $v \in W$. $\square$

Appendix A. Equivalence in terms of convergence

We defined equivalence of two norms on a vector space V in terms of them defining the same open subsets of V. Equivalence of norms can also be characterized in terms of having the same convergent sequences and limits.

Theorem A.1. *Let $(K, |\cdot|)$ be a valued field. The following properties of norms $\|\cdot\|$ and $\|\cdot\|'$ on a K-vector space V are equivalent to each other.*

1. *The open subsets of V defined by $\|\cdot\|$ and $\|\cdot\|'$ are the same.*
2. *The convergent sequences and their limits in V defined by $\|\cdot\|$ and $\|\cdot\|'$ are the same.*

A more general result applies to two metrics on a set, not just two norms on a vector space.

Theorem A.2. *The following properties of metrics d and d' on a set X are equivalent to each other.*

(1) *The open subsets of X defined by d and d' are the same.*
(2) *The convergent sequences and their limits in X defined by d and d' are the same.*

When $X = V$ is a vector space and the two chosen metrics on V are induced by norms on V, then Theorem A.2 becomes Theorem A.1.

Proof. (1) $\Longrightarrow$ (2): Convergence in X relative to a metric can be described in terms of open subsets of X for that metric: to say $d(x_n, x) \to 0$ as $n \to \infty$ means for every open subset $U \subset X$ relative to d such that $x \in U$, there is an $N \geq 1$ such that $n \geq N \Longrightarrow x_n \in U$. By (1), a subset of X that is open relative to d' is also dopen relative to d, so for every open subset $U \subset X$ relative to d' such that $x \in U$, there is an $N \geq 1$ such that $n \geq N \Longrightarrow x_n \in U$. Thus $x_n \to x$ relative to the topology defined by d', which means $d'(x_n, x) \to 0$. Since (1) is symmetric for d and d', a similar argument shows each convergent sequence in X relative to d' is convergent relative to d with the same limit.

(2) $\Longrightarrow$ (1): Since open subsets are complements of closed subsets, proving (1) is equivalent to proving the closed subsets of X defined by d and d' are the same. Let $C \subset X$ be closed relative to d. To show C is closed relative to d', let $\{x_n\}$ be a sequence in C that converges to some $x \in X$ relative to d': $d'(x_n, x) \to 0$. By hypothesis, $d(x_n, x) \to 0$, so $x \in C$ since C is closed relative to d. Hence every sequence in C that converges in X relative to d' has its d'-limit in C, so C is closed relative to d'.

The proof that every closed subset of X relative to d' is closed relative to d is the same, with the roles of d and d' exchanged. $\square$

Theorem A.2 does not generalize directly to topological spaces: two different topologies (even Hausdorff) could have the same convergent sequences (with the same limits). See `https://math.stackexchange.com/questions/76691`. The issue is that *sequences* may not be enough to capture everything about a topology. But if we use nets, which are a generalization of sequences, then we can say a topology is determined by its convergent nets (and their limits). See `https://mathoverflow.net/questions/19285`.