KLEIN'S EVANSTON LECTURES.

The Evanston Colloquium: Lectures on mathematics, delivered from Aug. 28 to Sept. 9, 1893, before members of the Congress of Mathematics held in connection with the World's Fair in Chicago, at Northwestern University, Evanston, Ill., by FELIX KLEIN. Reported by Alexander Ziwet. New York, Macmillan, 1894. 8vo. x and 109 pp.

THIS little volume occupies a somewhat unique position in mathematical literature. Even the *Commission permanente* would find it difficult to classify it and would have to attach a bewildering series of symbols to characterize its contents. It is stated as the object of these lectures "to pass in review some of the principal phases of the most recent development of mathematical thought in Germany"; and surely, no one could be more competent to do this than Professor Felix Klein. His intimate personal connection with this development is evidenced alike by the long array of his own works and papers, and by those of the numerous pupils and followers he has inspired. But perhaps even more than on this account is he fitted for this task by the well-known comprehensiveness of his knowledge and the breadth of view so characteristic of all his work.

In these lectures there is little strictly mathematical reasoning, but a great deal of information and expert comment on the most advanced work done in pure mathematics during the last twenty-five years. Happily this is given with such freshness and vigor of style as makes the reading a recreation. As the preface tells us, the twelve lectures here reproduced were given before twenty or more mathematicians from American colleges and universities, during the two weeks following the Chicago Congress of Mathematics last summer. Professor Klein had attended the Congress as official commissioner from Germany, and had contributed much to the success of that gathering. With the eager co-operation of a majority of the members of the Congress, he arranged the Colloquium at Evanston. This by no means easy task he undertook solely from love of his chosen science, and in the hope that thus he might contribute to the promotion of research in America in those parts of pure mathematics which at present engage the attention of the foremost European investigators. The noteworthy list of interested auditors prognosticates the realization of this hope. It is to the worker in special fields, and to the ambitious student, that these lectures will prove most useful. A complete review being in the nature of the case impossible, we shall present only briefly points of special interest.

First is discussed the geometrical work of Clebsch, it being presumed that his contributions to the theory of algebraic forms are too generally known in this country to require notice. Regarding as his most secure claim to remembrance the union that he effected between Riemann's theory of Abelian functions and the discoveries of Cayley, Sylvester, and Aronhold relating to algebraic curves, Klein states two objections to the presentation of this union given in Clebsch and Gordan's book. (1) It assumes without proof as the most general algebraic curves those that have no higher singularities than crunodes and acnodes; and (2) by employing the curve as the fundamental thing, to the exclusion of the equivalent Riemann surface, it renders obscure some points that by the other means would become self-evident, e.g., the invariant character of the deficiency-number p. The extension of the methods of the curve theory by Clebsch and his followers to the treatment of surfaces and of differential equations is emphasized as opening a rich and not too remote field of inquiry.

Two parts of Sophus Lie's closely connected researches come in for extended notice,—his sphere-geometry and his theory of "contact-transformations" as applied to linear differential equations. The former is shown to correspond to the line-geometry of Plücker, tangent spheres being correlated to intersecting lines. We quote from p. 17: "Perhaps the most striking example of the fruitfulness of this work of Lie's is his discovery that by means of this transformation the lines of curvature of a surface are transformed into asymptotic lines of the transformed surface, and *vice versa*. This appears by taking the definition given above for the lines of curvature and translating it word for word into the language of line-geometry. Two problems in the infinitesimal geometry of surfaces, that had long been regarded as entirely distinct, are thus shown to be really identical. This must certainly be regarded as one of the most elegant contributions to differential geometry made in recent times."

From most original thinkers we look for the more lucid and complete exposition of their theories in their later publications. One is therefore surprised at the following hint (p. 9): "To fully understand the mathematical genius of Sophus Lie, one must not turn to the books recently published by him in collaboration with Dr. Engel, but to his earlier memoirs, written during the first years of his scientific career. There Lie shows himself the true geometer that he is," etc. No doubt the contrary advice would be preferable for those whose bent is rather to analysis than to geometry. We venture the opinion that most students would find in Klein's own latest lecture-course (Höhere Geometrie II, lithographed, Göttingen, 1893) the best possible introduction to Lie's works.

Two lectures on recent papers of his own are full of germinal ideas, those on the real shape of algebraic curves and surfaces, and on theory of functions and geometry. The former brings to light certain problems most open to attack, and offers aid from the theory of symmetric Riemann surfaces; the latter presents in outline an ingenious application of geometric intuition to an analytical problem. So simple a thing as the structure of a multiply covered spherical triangle is made to indicate the number of zeros of a hypergeometric function between given limits.

On the subject of the validity and degree of exactness of mathematical definitions and axioms we find mainly a résumé of his paper in the *Math. Annalen*, vol. 22, with the added interest of several apt illustrations. To show the unreliability of the "naïve intuition," a curve is defined by purely geometric terms, which proves to have indeed a tangent at every point, but nowhere a determinate curvature! The conclusions reached are made a text for urging upon our schools the combination, in instruction, of the applied sciences with the abstract. "Separation could only be deplored, for it would necessarily be followed by shallowness on the side of the applied sciences, and by isolation on the part of pure mathematics."

The famous proofs by Hermite, Lindemann, and Weierstrass, of the transcendental character of the numbers e and π, previously inaccessible to most college students, have been replaced within the past year by far simpler proofs by Hilbert, Hurwitz, and Gordan; their papers are now collected in vol. 43 of the *Math. Annalen*. A lecture upon Hilbert's proof states that "The problem has thus been reduced to such simple terms that the proofs for the transcendency of e and π should henceforth be introduced into university teaching everywhere."

The views presented on the solution of higher algebraic equations will not be novel to readers of the *Ikosaeder*, but are perhaps even more sharply and forcibly stated here. In opposition to a current restriction of the "solution of an equation" to mean its solution by radicals, Klein insists that solution shall denote the reduction of an equation to one of certain definite *normal equations*. Having accomplished this for equations of degree 5, 6, and 7, he formulates the general problem as follows, first bringing into one class all equations whose Galois groups are isomorphic: "Among the problems having isomorphic groups we consider as the simplest the one that has the least number of variables, and call this the *normal problem*. This problem must be considered as solvable by series of any kind. The question is, to reduce the other isomorphic problems to the normal problem." For the general equation of the eighth degree no normal problem of lower order has yet been found.

Topics which we can only name here are ideal numbers, recent advances in hyperelliptic and Abelian functions, and most recent researches in non-Euclidean geometry. The terms *line-lattice* and *point-lattice* strike us as felicitous. We learn that a space of zero-curvature is not necessarily infinite; and that our geometrical demonstrations "have no absolute objective truth, but are true only for the present state of our knowledge." For "we can never tell whether an enlarged conception may not lead to further possibilities that would have to be taken into account. From this point of view we are led in geometry to a certain modesty, such as is always in place in the physical sciences."

The closing lecture is devoted to a description of methods adopted in the mathematical department at Göttingen and to some sound advice to students intending to go to Germany for the purpose of studying mathematics. "As regards my own higher lectures," says Prof. Klein, "I have pursued a certain plan in selecting the subjects for different years, my general aim being *to gain, in the course of time, a complete view of the whole field of modern mathematics, with particular regard to the intuitional or* (in the highest sense of the term) *geometrical standpoint.* This general tendency you will, I trust, also find expressed in this colloquium, in which I have tried to present, within certain limits, a general programme of my individual work."

The book contains as an appendix a translation (by Prof. H. W. Tyler) of Klein's sketch "The development of mathematics at the German universities," written originally for the Exposition volume "Die deutschen Universitäten." This brief historical essay serves as an introduction to the lectures in so far as it carries the history up to the year 1870, while the lectures are devoted to the last quarter of a century.

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