THE INDEPENDENCE OF THE AXIOM OF CHOICE

by

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§1. Introduction

Ever since its introduction into mathematics the Axiom of Choice has been regarded as a principle which in some sense is not as intuitively obvious as the other axioms of set theory. Many mathematicians have gone so far as to reject its use altogether, and in some cases great pains have been taken in order to avoid using it. In his fundamental papers, Gödel [2],[3], showed that if one is willing to accept the fact that no contradiction can be obtained using the other axioms, then no contradiction can be obtained by using the Axiom of Choice. This result is sometimes referred to as the Consistency of the Axiom of Choice, though more correctly it should be called the relative consistency since the consistency of set theory itself cannot be proved without appealing to more powerful mathematical ideas than set theory itself. Though quite complicated, Gödel's proof of the relative consistency can be written down in elementary number theory.

There is another aspect to this question which is not however answered by Gödel's work and that is whether there is any need at all to adjoin the Axiom of Choice as a separate axiom or whether it already is a consequence of the other axioms. This is the question of independence. There is, of course, another famous example of a question of independence

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which occupied mathematicians for quite a time, namely the parallel postulate in Geometry. Here the question of consistency did not arise since Euclidean Geometry was well known and was on a firm foundation. The discovery of Non-Euclidean Geometry and therefore the independence of this axiom, only came after many unsuccessful attempts to prove it. In the case of the Axiom of Choice, considerable effort was devoted to finding more acceptable formulations of it, but it seems to have been recognized almost immediately as a genuinely new principle. No proof however was found of its independence, though certain partial results were obtained. These consisted primarily of weakening or omitting entirely one of the other axioms. This enabled one to construct models for set theory in which some form of the Axiom of Choice was violated, although these examples were always unnatural in the sense that the "sets" of the model were not intuitively sets. A good discussion of these results can be found in [1], [4] [5][6]. We merely mention that the basic principle involved was either directly to introduce individuals $x, y, \ldots$ which have no members and yet are distinct, or to do this in an indirect manner by constructing infinite chains e.g. x_n such that $x_{n+1} = \{x_n\}$. Clearly if $y_{n+1} = \{y_n\}$, then x_1 and y_1 cannot be truly distinguished so that one may, by further argument, violate the Axiom of Choice.

In the present paper we show the independence of the Axiom of Choice and the Continuum Hypothesis from the full set of axioms for set theory, i.e. either Zermelo-Fraenkel or Gödel-Bernays set theory. The form of the proof is such that it will extend even if we adjoin axioms of a "natural type" to set theory. Also the models we construct will always be collections of sets and ϵ will always have its usual meaning. We rely heavily on the ideas which were developed in [3] and since this work has become a rather standard reference, we try to follow the notation used there. However, it did seem more natural to us to use the Zermelo-Fraenkel (Z-F) formulation rather than the Gödel-Bernays formulation. Also a rather interesting result concerning the independence of the Continuum Hypothesis from the Axiom of Choice is given. Of course, there are many special applications of the Axiom of Choice in mathematics, and the question is now open to determine which of these is independent of Z-F theory. For example, it would be interesting to know if it is consistent to assume that all sets of real numbers are measurable.

§2. In general we follow the notation of [3], so that we sometimes write (x) in place of $\forall x$. Also $\exists !$ means there exists a unique element. We first state the axioms for Z-F theory. These axioms speak about undefined objects called sets, and an undefined relation ϵ , called membership.

Axiom 1. Extensionality

$$(x)(y) \quad x = y \Longleftrightarrow (z)(z \in x \Longleftrightarrow z \in y) .$$

Axiom 2. Null set

$$\exists x \quad (y)(\neg\, y \in x) .$$

Axiom 3. Unordered pairs

$$(x)(y) \; \exists z \; (t)(t \in z \Longleftrightarrow t = x \vee t = y) .$$

Axiom 4. Power set

$$(x) \; \exists y \; (t)(t \in y \Longleftrightarrow t \subseteq x) .$$

Here $t \subseteq x$ means $(u)(u \in t \Longrightarrow u \in x)$.

Axiom 5. Infinity

$$\exists x(\emptyset \in x \,\&\, (y)(y \in x \Longrightarrow \{y\} \in x)) \,.$$

Axiom 6_n. Replacement

This axiom is a countable collection of axioms, one for each proposition $\mathcal{O}\!\mathcal{l}_n(x_1, \ldots x_k)$ which can be constructed from symbols for variables, propositional connectives and quantifiers, having precisely $x_1, \ldots, x_k$ as unbounded variables.

$$(x_3, \ldots ,x_k)(y)\exists z[(x_1)\exists!\, x_2\, \mathcal{O}\!\mathcal{l}_n(x_1, \ldots ,x_k) \Longrightarrow (t)t \in z \Longleftrightarrow \exists u(u \in y \,\&\, \mathcal{O}\!\mathcal{l}_n(u,t,x_3, \ldots ,x_k))] \,.$$

The meaning of 6_n is that if $\mathcal{O}\!\mathcal{l}(x_1,x_2)$ is a relation which for x_1 has a unique x_2 satisfying, i.e. defines x_2 as a single-valued function of x_1, then the range of this function when restricted to a set y is a set z. Most sets in mathematics are actually defined by 6_n e.g. the set of primes can be defined using the relation $\mathcal{O}\!\mathcal{l}(x_1,x_2) \equiv$ either x_1 is not an integer and $x_2 = 0$ or $x_1 = x_2$ and x_1 is a prime.

Axiom 7. Sum-Set

$$(x)\exists y(z)(z \in y \Longleftrightarrow \exists t(t \in x \,\&\, z \in t)) \,.$$

Axiom 8. Regularity

$$(x)\exists y\, (x = \emptyset \vee (y \in x \,\&\, (z)(z \in x \Longrightarrow \neg\, z \in y))) \,.$$

This is to say that every set contains a minimal element with respect to

ϵ (but not with respect to $\subset$).

We are now ready to state the main theorem. Here ω denotes the set of integers. We repeat our assertion that all models for Z-F are models of sets with ϵ in the ordinary sense.

<u>Theorem 1</u>:

There are models for Z-F in which the following occur

1. There is a set $\underline{a}$, $\underline{a} \subset \omega$ such that $\underline{a}$ is not constructible in the sense of [3] , yet the Axiom of Choice and the Generalized Continuum Hypothesis both hold.

2. The continuum (i.e. $\mathcal{P}(\omega)$ where $\mathcal{P}$ means power set) has no well-ordering.

3. The Axiom of Choice holds, but $\aleph_1 \neq 2^{\aleph_0}$.

4. The Axiom of Choice for countable pairs of elements in $\mathcal{P}(\mathcal{P}(\omega))$ fails.

Note that 4 implies that there is no simple ordering of $\mathcal{P}(\mathcal{P}(\omega))$. Since the Axiom of Constructibility implies the Generalized Continuum Hypothesis [3], and the latter implies the Axiom of Choice [7], Theorem 1 completely settles the question of the relative strength of these axioms.

We recall the definition of Constructibility. If $0 \leq i \leq 8$, and α and β represent ordinals, there is a natural well-ordering on triples $\langle i,\alpha,\beta\rangle$, denoted by S , given in [3] p. 36 Dfn. 9.2. Let J denote the unique order preserving map of the set of these triples onto the set of all ordinals.

<u>Definition 1.</u>

If $J(\langle i,\alpha,\beta\rangle) = \gamma$ we put $N(\gamma) = i$, $K_1(\gamma) = \alpha$, $K_2(\gamma) = \beta$.

Then one defines the sets F_α, by transfinite induction:

<u>Definition 2.</u>

$$\text{If } N(\alpha) = 0 \quad , \quad F_\alpha = \{F_\beta | \beta < \alpha\} \; .$$
$$\text{If } N(\alpha) = i \; , \quad i > 0 \; , \quad F_\alpha = \mathcal{F}_i(K_1(\alpha), \; K_2(\alpha))$$

where $\mathcal{F}_i$ are defined on p. 35 [3]. The eight operation $\mathcal{F}_i$ are the fundamental operations by means of which all formulas are defined.

<u>Axiom of Constructibility</u>:

$(x) \exists \alpha \, (F_\alpha = x)$. We say x is constructible if $\exists \alpha (x = F_\alpha)$. The idea of classifying sets by means of the ordinals needed to construct them is fundamental for all that follows.

§3. <u>Axiom of Constructibility</u>

This section is devoted to proving statement 1 of Theorem 1. <u>For the remainder of this paper $\mathfrak{M}$ will denote a fixed countable collection of sets which is a model for Z-F theory and $V = L$ (Axiom of Constructibility) and such that $x \in \mathfrak{M}, \, y \in x \Longrightarrow y \in \mathfrak{M}$.</u> The purpose of the last condition is to insure that there are no "extraneous" sets involved in $\mathfrak{M}$ and if α is an ordinal $\alpha = \text{rank } \alpha$. Let <u>a</u> denote a subset of ω, which is <u>not necessarily</u> in $\mathfrak{M}$. Then we define sets $F_\alpha(a)$ in analogy with Definition 2 as the set constructed at ordinal α starting from a. More precisely:

<u>Definition 3.</u>

Let S be the well-ordering of [3]. Let J now denote the unique order-preserving map of triples $\langle i,\alpha,\beta \rangle$, $0 \leq i \leq 8$, α,β ordinals in $\mathfrak{M}$ onto the ordinals in $\mathfrak{M}$ which are $\geq \omega+2$. (Note that the integers in $\mathfrak{M}$ are absolute). Let N, K_1, K_2 be as in Definition 1. For $0 \leq \alpha \leq \omega$, let $F_\alpha(a) = \alpha$, and let $F_{\omega+1}(a) = a$. For $\alpha \geq \omega+2$, define

$F_\alpha(a)$ as in Definition 2. Finally, put $\mathcal{N} = \{F_\alpha(a) \mid \alpha \in \mathcal{M}\}$. We note that by 9.25 [3], $K_1(\alpha) \leq \alpha$, $K_2(\alpha) \leq \alpha$ and strict inequality holds if $N(\alpha) > 0$.

We note that each $F_\alpha(a)$ is a collection of $F_\beta(a)$ for some $\beta < \alpha$. Since $\underline{a}$ was not necessarily in $\mathcal{M}$, it is not in general true that $\mathcal{N}$ is a model for Z-F theory. Since $\mathcal{M}$ is countable and so has only countably many α, it is clear that there exist $a \subseteq \omega$ such that $\underline{a}$ is not constructible with respect to the ordinals in $\mathcal{M}$. Our problem is to find such an $\underline{a}$ with the additional property that $\mathcal{N}$ is a model for Z-F.

Rather than immediately determine such an $\underline{a}$ directly, we begin by asking questions about the possible relations in $\mathcal{N}$ which may be satisfied. Thus, we shall now write F_α as a purely formal symbol where α is always assumed to range over ordinals in $\mathcal{M}$. We shall often write α in place of F_α if $0 \leq \alpha \leq \omega$ and $\underline{a}$ in place of $F_{\omega+1}$. We also point out the very important fact that the ordinals in $\mathcal{N}$ are precisely those in $\mathcal{M}$ by virtue of the obvious fact that rank $F_\alpha(a) \leq \alpha$. Conversely, one can easily see that for each $\alpha \in \mathcal{M}$, $\exists \alpha' \in \mathcal{M}$ such that $\alpha = F_{\alpha'}(a)$ independently of $\underline{a}$.

Definition 4.

A formula is defined by

1) $x \in y$, $F_\alpha \in x$, $x \in F_\alpha$, $F_\alpha \in F_\beta$ are formulas where x and y are variables.

2) if φ and ψ are formulas, so are $\neg \varphi$ and $\varphi \,\&\, \varphi$.

3) only 1) and 2) define formulas.

Definition 5.

A Limited Statement is a formula $\mathcal{O}\!\mathcal{L}(x_1, \ldots, x_n)$ with n quantifiers placed in front each of the form $(x_i)_\alpha$ or $\exists_\alpha x$ where α is an ordinal in $\mathcal{M}$.

Our intention is that if $T_\alpha = \{F_\beta | \beta < \alpha\}$, the quantifier $(x)_\alpha$ shall intuitively mean "for all x in T_α" and similarly for $\exists_\alpha x$.

Definition 6.

An Unlimited Statement is a formula $\mathcal{O}\!\!\mathcal{L}(x_1, \ldots, x_n)$ with n quantifiers placed in front each of the form (x_i) or $\exists x_i$.

Since the notion of construction can be formalized it is intuitively clear how each Limited Statement can be made to correspond to an Unlimited Statement.

Definition 7.

P will always denote a finite set of conditions $m_k \in a$ and $\neg n_\ell \in a$, where no m_k equals any n_ℓ, these being of course integers. Formally, P is merely an ordered pair of disjoint finite subsets of ω.

Definition 8.

The rank of a Limited Statement $\mathcal{O}\!\!\mathcal{L}$ is said to be (α, r) if the number of quantifiers is r and if α is the least ordinal strictly greater than all β such that F_β occurs in $\mathcal{O}\!\!\mathcal{L}$ and greater than equal to all β such that $(x)_\beta$ or $\exists_\beta x$ occurs in $\mathcal{O}\!\!\mathcal{L}$. (This means that intuitively $\mathcal{O}\!\!\mathcal{L}$ is a statement in T_α). We say $(\alpha, r) \prec (\alpha', r')$ if $\alpha < \alpha'$ or $\alpha = \alpha'$, $r < r'$.

It is clear that all our statements are in so-called prenex form. It is obvious how to form the negation, conjunction etc. of such statements and still stay in prenex form. Thus we shall not always bother to write our statements in such form, if it is clear what the corresponding prenex form would be.

The key point for all that follows is contained within the following:

Definition 9. First Fundamental Definition

Let $\mathfrak{A}$ be a Limited Statement, P a set of conditions as in Definition 7. By transfinite induction on the rank of $\mathfrak{A}$ we shall define what we mean by saying "P forces $\mathfrak{A}$ ".

I. If $r > 0$, we say P forces $\mathfrak{A} \equiv (x)_\alpha \mathfrak{b}(x)$, if for all $P' \supseteq P$, P' does not force $\neg \mathfrak{b}(F_\beta)$ for any $\beta < \alpha$.

II. If $r > 0$, we say P forces $\mathfrak{A} \equiv \exists_\alpha x \, \mathfrak{b}(x)$, if for some $\beta < \alpha$ P forces $\mathfrak{b}(F_\beta)$.

III. If $r = 0$, P forces $\mathfrak{A}$ if $F_{\alpha_i} \in F_{\beta_i}$ are the components out of which $\mathfrak{A}$ is formed by means of propositional connectives, and for each i, either $F_{\alpha_i} \in F_{\beta_i}$ or $\neg F_{\alpha_i} \in F_{\beta_i}$ is forced in such a manner that in the usual sense of the propositional calculus, $\mathfrak{A}$ is true.

IV. If $r = 0$, and $\mathfrak{A}$ is of the form $F_\alpha \in F_\beta$, then

i) if $\alpha, \beta, \leq \omega$ $\mathfrak{A}$ is forced if the obvious relation between α and β hold.

ii) if P is the condition $m_k \in a$, $\neg n_\ell \in a$, then P forces these statements i.e. $F_{m_k} \in F_{\omega+1}$, $\neg F_{n_\ell} \in F_{\omega+1}$.

iii) P forces $\neg \omega \in a$ i.e. $\neg F_\omega \in F_{\omega+1}$.

iv) P forces $\neg F_\alpha \in F_\alpha$ for all α.

v) if $\alpha < \beta$, $\beta > \omega+1$, then $\mathfrak{A}$ is forced if and only if the corresponding statement in T_β formed by going back to the definition of F_β is forced. We make this more precise by enumerating each of the nine possibilities, $0 \leq i \leq 8$ and merely translating the definition of F_β as given on p.37 [3]. To save space we shall write $x = y$ as an abbreviation for $(z)_\beta(z \in x \Longleftrightarrow z \in y)$. Also put $K_1(\beta) = \gamma$, $K_2(\beta) = \delta$.

0.) if $N(\beta) = 0$, P forces $F_\alpha \in F_\beta$ (recalling that throughout (v) we have $\alpha < \beta$)

1.) if $N(\beta) = 1$, P forces $F_\alpha \in F_\beta$ if P forces $F_\alpha = F_\gamma \vee F_\alpha = F_\delta$.

2.) if $N(\beta) = 2$, P forces $F_\alpha \in F_\beta$ if P forces

$$\exists_\beta x \exists_\beta y(F_\alpha = \langle x,y\rangle \ \& \ x \in y \ \& \ F_\alpha \in F_\gamma) .$$

3.) if $N(\beta) = 3$, P forces $F_\alpha \in F_\beta$ if P forces $F_\alpha \in F_\gamma \ \& \ \neg F_\alpha \in F_\delta$.

4.) if $N(\beta) = 4$, P forces $F_\alpha \in F_\beta$ if P forces

$$\exists_\beta x \exists_\beta y(F_\alpha = \langle x,y\rangle \ \& \ F_\alpha \in F_\gamma \ \& \ y \in F_\delta) .$$

5.) if $N(\beta) = 5$, P forces $F_\alpha \in F_\beta$ if P forces

$$\exists_\beta x(F_\alpha \in F_\gamma \ \& \ \langle x,F_\alpha\rangle \in F_\delta) .$$

6.) if $N(\beta) = 6$, P forces $F_\alpha \in F_\beta$ if P forces

$$\exists_\beta x \exists_\beta y(F_\alpha \in F_\gamma \ \& \ F_\alpha = \langle x,y\rangle \ \& \ \langle y,x\rangle \in F_\delta) .$$

7.) if $N(\beta) = 7$, P forces $F_\alpha \in F_\beta$ if P forces

$$\exists_\beta x \exists_\beta y \exists_\beta z(F_\alpha = \langle x,y,z\rangle \ \& \ F_\alpha \in F_\gamma \ \& \ \langle y,z,x\rangle \in F_\delta) .$$

8.) if $N(\beta) = 8$, P forces $F_\alpha \in F_\beta$ if P forces

$$\exists_\beta \exists_\beta y \exists_\beta z(F_\alpha = \langle x,y,z\rangle \ \& \ F_\alpha \in F_\gamma \& \ \langle x,y,z\rangle \in F_\delta) .$$

vi) If $\mathfrak{A}$ is of the form $\neg F_\alpha \in F_\beta$, $\alpha < \beta$, then P forces $\mathfrak{A}$ if $N(\beta) = i > 0$ and P forces the negation of the corresponding statements in v).

vii) if $\mathfrak{A}$ is of the form $F_\alpha \in F_\beta$ where $\alpha > \beta$, P forces $\mathfrak{A}$ if P forces the statement $\exists_\beta x(F_\alpha = x \ \& \ x \in F_\beta)$ where the formula

$F_\alpha = x$ is replaced by $(y)_\alpha (y \in F_\alpha \Longleftrightarrow y \in x)$ and $y \in F_\alpha$ in turn is replaced by the corresponding statement in v) which is justified since $y \in T_\alpha$.

viii) if $\mathfrak{A}$ is of the form $\neg F_\alpha \in F_\beta$ and $\alpha > \beta$, then P forces $\mathfrak{A}$ if it forces the negation of the corresponding statement in vii).

Definition 10. Second Fundamental Definition

We say P forces $(x)\mathfrak{A}(x)$, $\mathfrak{A}(x)$ unlimited, if for all $P' \supset P$, P' does not force $\neg \mathfrak{A}(F_\alpha)$ for any α.

We say P forces $\exists x \mathfrak{A}(x)$, $\mathfrak{A}(x)$ unlimited, if for some α, P forces $\mathfrak{A}(F_\alpha)$.

The set of all Limited Statements can be put into one-one correspondence with the ordinals of $\mathfrak{M}$ in a natural and straightforward manner. Once having done this, it follows that the relation "P forces $\mathfrak{A}_\alpha$" where $\mathfrak{A}_\alpha$ is the Limited Statement corresponding to α, can be formulated in Z-F theory. This is because the Fundamental Definition of forcing is by Transfinite Induction and all operations in the definition can be formalized as simple operations using the axioms of Z-F theory. It is also true that one may list all Unlimited Statements $\mathfrak{A}_\alpha$ but the relation "P forces $\mathfrak{A}_\alpha$" is not formalizable (as a relation between α and P) in Z-F since one has to know in def. 10 whether for <u>any</u> β, say P forces $\mathfrak{A}(\beta)$ which means intersecting the universe with a suitable "class", which cannot be done in Z-F theory. Of course, for any particular Unlimited Statement $\mathfrak{A}$, the notion of "P forcing $\mathfrak{A}$" can be formalized since this requires using the Axiom of Replacement only a finite number of times. It would be extremely tedious to actually give the sentence in the formal language language expressing the relation "P forces the α^{th} Limited Statement" as well as the various functions giving the ordinal β such that $\mathfrak{A}_\alpha$ becomes $\mathfrak{A}_\beta$ when F_γ is substituted for the first variable, say, although in principle this can be done. The above remarks are extremely important and are the cornerstone of the whole construction since ultimately the set $\underline{a}$ will be in some vague sense a limit of "generic" subsets of integers in $\mathfrak{M}$, which in turn will account for why $\mathfrak{N}$ is already a model for Z-F without further extension.

Lemma 1: P does not force $\mathfrak{A}$ and $\neg\mathfrak{A}$, for any $\mathfrak{A}$ and P.

Proof:

Let $\mathfrak{A}$ be a Limited Statement. We prove the Lemma by induction on the rank of $\mathfrak{A}$. If $r = 0$, if we assume that each component $F_\alpha \in F_\beta$ is forced uniquely to be either true or false, the result clearly follows for $\mathfrak{A}$ itself. Since P forcing $F_\alpha \in F_\beta$ or $\neg F_\alpha \in F_\beta$ was thrown back to a statement in T_α or its corresponding negation, by induction the result is true for $r = 0$, if we observe that it trivially holds if $\alpha, \beta \leq \omega+1$. If $r > 0$, then if P forces $\exists x_\alpha \mathfrak{A}(x)$ and $(x)_\alpha \neg\mathfrak{A}(x)$, then P must force $\mathfrak{A}(F_\beta)$ for $\beta < \alpha$, which violates the definition of P forcing $(x)_\alpha \neg\mathfrak{A}(x)$. Thus the Lemma is true in this case. Finally, if $\mathfrak{A}$ is unlimited the Lemma follows in the same manner by induction on the number of quantifiers.

Lemma 2: If P forces $\mathfrak{A}$ and $P' \supseteq P$, then P' forces $\mathfrak{A}$.

Proof:

As in the proof of Lemma 1 we need only discuss the case of P forcing $(x)_\alpha \mathfrak{A}(x)$ or $\exists_\alpha x\, \mathfrak{A}(x)$ as well as the corresponding unlimited statements. We proceed by induction on the rank. If P forces $(x)_\alpha \mathfrak{A}(x)$ and $P' \supseteq P$, then if P' did not force the same statement, we would have that for some $P'' \supseteq P'$, P'' forces $\neg \mathcal{O}\!\ell(F_\beta)$ for some $\beta < \alpha$. This contradicts the hypothesis since $P'' \supseteq P$. If P forces $\exists_\alpha x \mathcal{O}\!\ell(x)$, then it must force $\mathcal{O}\!\ell(F_\beta)$ for some $\beta < \alpha$. If $P' \supseteq P$, by induction we assume P' forces $\mathcal{O}\!\ell(F_\beta)$ so that P' forces $\exists_\alpha x$ (x). Similarly for unlimited statements.

Lemma 3: For any statement $\mathcal{O}\!\ell$, and conditions P, there is $P' \supseteq P$ such that either P' forces $\mathcal{O}\!\ell$ or P' forces $\neg \mathcal{O}\!\ell$.

Proof:

If $\mathcal{O}\!\ell$ has $r = 0$ and has components $F_\alpha \in F_\beta$ with $\alpha, \beta \leq \omega+1$, then it is clear that any statement $n \in a$ or its negation may be forced by extending P, whereas the other cases are already forced. If $r = 0$ in general the conclusion follows if we can prove it for the components. We are thus left with the case $r > 0$. If P does not force $(x)_\alpha \mathcal{O}\!\ell(x)$, then there is a $P' \supseteq P$ such that P' forces $\neg \mathcal{O}\!\ell(F_\beta)$, $\beta < \alpha$, which means P' forces the negation. Similarly for unlimited statements.

Definition 11. The definition of the model $\mathcal{N}$

Let α_n be an enumeration of the ordinals in $\mathcal{M}$, (this enumeration is of course not expressible in $\mathcal{M}$), and let $\mathcal{O}\!\ell_n$ be an enumeration of all statements, limited and unlimited, built up from the F_α and the logical symbols. Let P_o be empty. Let P_{2n+1} be the first (in any natural ordering) extension of P_{2n} which forces either $\mathcal{O}\!\ell_n$ or $\neg \mathcal{O}\!\ell_n$. If k is the first integer not appearing in P_{2n-1}, form P_{2n} by adjoining the condition $k \in a$ or $\neg k \in a$ to P_{2n-1} according to whether k does not or does belong to the set constructed (in the sense of [3]) by the ordinal α_n. Since every statement $k \in a$ appears among the $\mathcal{O}\!\ell_n$, we may put

$a = \lim_{n\to\infty} P_n$ and we set $\mathcal{N} = \{F_\alpha(a) \mid \alpha \in \mathcal{M}\}$.

Theorem 2.

$\mathcal{N}$ is a model for Z-F , and the set a is not constructible in $\mathcal{N}$. In $\mathcal{N}$ the Axiom of Choice and Generalized Continuum Hypothesis hold.

The proof of Theorem 2 will require several lemmas. We note that the second statement in Theorem 2 can be easily verified since in $\mathcal{N}$ every set is constructible from a , so that we have a natural well-ordering of all of $\mathcal{N}$ by saying $F_\alpha(a) \prec F_\beta(a)$ if $\alpha < \beta$. Also the argument given in [3] Chapter VIII carries over with almost no change.

It is clear how each unlimited statement involving the formal symbols F_α becomes an actual statement about $\mathcal{N}$. As remarked above, this can also be said about limited statements since the construction can be formalized in $\mathcal{N}$. However, for the purpose of Lemma 4 below we shall interpret the quantifiers $(x)_\alpha$ and $\exists_\alpha x$ to merely mean that x is restricted to lie in T_α.

Lemma 4: All statements in $\mathcal{N}$ which are forced by some P_n are true in $\mathcal{N}$ and conversely.

Proof:

We proceed by induction on the rank of limited statements. It suffices to discuss $r > 0$, since the other reductions follow almost immediately from definition 9. If P_n forces $(x)_\alpha \mathcal{O}(x)$, then if $\beta < \alpha$, no P_n forces $\neg \mathcal{O}(F_\beta)$. By induction we may assume $\mathcal{O}(F_\beta)$ is true in $\mathcal{N}$, so that $(x)_\alpha \mathcal{O}(x)$ is therefore true in $\mathcal{N}$. If P_n forces $\exists_\alpha \mathcal{O}(x)$, then P_n must force $\mathcal{O}(F_\beta)$ for some $\beta < \alpha$, which by induction means $\mathcal{O}(F_\beta)$ is true in $\mathcal{N}$, so that therefore $\exists_\alpha x\, \mathcal{O}(x)$ is true in $\mathcal{N}$.

Similarly for unlimited statements. Since every statement or its negation is forced eventually, the converse is also true.

Our next Lemma is a precise statement of the general principle stated after definition 10.

Lemma 5: Let $\mathfrak{A}(x,y)$ be a fixed unlimited statement. There exists a statement $\Phi_{\mathfrak{A}}(P,\alpha,\beta)$ which asserts that P forces $\mathfrak{A}(F_\alpha,F_\beta)$ and for all $\gamma < \beta$, P does not force $\mathfrak{A}(F_\alpha,F_\gamma)$.

If $\mathfrak{A}$ is an unlimited statement with r quantifiers, it is clear that r applications of the axiom of replacement will suffice to define $\Phi_{\mathfrak{A}}$ in terms of the corresponding notion for limited statements. Then, as mentioned before, for limited statements one may define a single relation which expresses the forcing condition.

Definition 12.

For an unlimited statement $\mathfrak{A}$ put $\Gamma_{\mathfrak{A}}(\alpha) = \sup\{\beta \mid \exists\, P, \alpha_1 < \alpha$ and $\Phi_{\mathfrak{A}}(P,\alpha_1,\beta)\}$.

Definition 12 is justified since $\Phi_{\mathfrak{A}}(P,\alpha,\beta)$ defines β uniquely in terms of P and α, if such a β exists. Thus, here we are using the replacement axiom in $\mathfrak{M}$, and we shall see that definition 12 is the key point in the verification of the replacement axiom in $\mathfrak{N}$.

Lemma 6: Let $\mathfrak{A}(x,y)$ be an unlimited statement such that for each x in $\mathfrak{N}$ there is a unique y such that $\mathfrak{A}(x,y)$. If α is an ordinal, and $C = \{F_\beta \mid \exists\, \alpha' < \alpha \;\&\; \mathfrak{A}(F_{\alpha'}, F_\beta)\}$, then $C \subseteq T_\gamma$ where $\gamma = \Gamma_{\mathfrak{A}}(\alpha)$.

Proof:

If $\mathfrak{A}(F_{\alpha'},F_\beta)$ is true in $\mathfrak{N}$, then it must be forced by some P_n. This clearly implies that $\beta < \Gamma_{\mathfrak{A}}(\alpha)$.

Lemma 7: Let $\mathcal{O}(x,y)$ be an unlimited statement of the form

$$Q_1 x_1 Q_2 x_2, \ \ldots \ , Q_n x_n \mathcal{B}(x,y,x_1, \ \ldots \ ,x_n)$$

where $\mathcal{B}$ has no quantifiers and Q_i are either existential or universal quantifiers. Assume $\mathcal{O}$ defines y as a single valued function of x. Then for each α there exist ordinals $\gamma_o, \ \ldots \ ,\gamma_n$ such that for $x \in T_\alpha$, there exist $y \in T_{\gamma_o}$ such that $\mathcal{O}(x,y)$ and for $\langle x,y\rangle$ in $T_\alpha \times T_{\gamma_o}$, the statement $\mathcal{O}(x,y)$ holds if and only if $\tilde{\mathcal{O}}(x,y)$ holds where $\tilde{\mathcal{O}}$ is the limited statement formed by restricting the quantifiers Q_i in $\mathcal{O}$ to range over T_{γ_i}.

Proof:

Lemma 6 implies the existence of γ_o. For $\langle x,y\rangle$ in $T_\alpha \times T_{\gamma_o}$ let $g_1(x,g,\delta)$ be the condition (i) if Q_1 is universal, that δ is the least ordinal such that $\neg Q_2 x_2, \ \ldots \ ,Q_n x_n \ (x,y,F_\delta,x_2, \ \ldots \ ,x_n)$ if such a δ exists, otherwise $\delta = \emptyset$ or (ii) if Q_1 is existential, that δ is the least ordinal such that $Q_2 x_2, \ \ldots \ ,Q_n x_n \mathcal{B}(x,y,F_\delta,x_2, \ \ldots \ ,x_n)$ if such a δ exists, otherwise $\delta = \emptyset$. By Lemma 6 we define

$$\gamma_1 = \sup\{\delta \mid \exists\langle x,y\rangle \ \text{ in } \ T_\alpha \times T_{\gamma_o} \ \text{ and } \ g_1(x,y,\delta)\}.$$

It follows then that for $\langle x,y\rangle$ in $T_\alpha \times T_{\gamma_o}$, $\mathcal{O}(x,y)$ is true if and only if $\mathcal{O}(x,y)$ with Q_1 restricted to range over T_{γ_1} is true. By induction define

$$\gamma_k = \sup\{\delta \mid \exists\langle x,y,x_1,\ldots,x_{k-1}\rangle \in T_\alpha \times T_{\gamma_o} \times \ldots \times T_{\gamma_{k-1}} \ \text{ and } \ g_k(x,y,x_1,\ldots,x_{k-1},\delta)\}$$

where $g_k(\cdot \ \cdot \ \cdot)$ is the condition (i) if Q_k is universal, that δ is the least ordinal such that $\neg Q_{k+1} x_{k+1},\ldots,Q_n x_n \mathcal{B}(x,g,x_1,\ldots,x_{k-1},F_\delta)$ if such a δ exists, otherwise $\delta = \emptyset$ or (ii) if Q_k is existential, that δ is the least ordinal such that $Q_{k+1}x_{k+1},\ldots,Q_n x_n \mathcal{B}(x,y,x_1,\ldots,x_{k-1},F_\delta)$ if such a δ exists, otherwise $\delta = \emptyset$. Lemma 7 then clearly follows.

<u>Lemma 8</u>: If $\mathcal{O}\!\mathcal{L}(x)$ is a limited statement (with one free variable), and $C = \{x | x \in T_\alpha \,\&\, \mathcal{O}\!\mathcal{L}(x)\}$ then for some β, $C = F_\beta$.

<u>Proof</u>:

This is an immediate consequence of the way F_α are defined, and of the fact that by means of $\mathcal{F}_i$ all statements may be built up, as well as the additional remark that since the construction of F_α may be formalized, each $T_\alpha = F_{\alpha'}$ for some α'.

<u>Lemma 9</u>: The Axiom of Replacement holds in $\mathcal{N}$.

<u>Proof</u>:

Let $\mathcal{O}\!\mathcal{L}(x,y)$ be an unlimited statement which defines y as a single valued function of x in $\mathcal{N}$. Let $z = F_\alpha$ be a set in $\mathcal{N}$. Let $\tilde{\mathcal{O}\!\mathcal{L}}(x,y)$ be as in Lemma 7. Consider the limited statement $\exists_\alpha x(x \in F_\alpha \,\&\, \tilde{\mathcal{O}\!\mathcal{L}}(x,y))$. The set of all y satisfying it and lying in T_{γ_0} is by Lemma 7 the same as the set $C = \{y | \exists x \in z \,\&\, \mathcal{O}\!\mathcal{L}(x,y)\}$, and by Lemma 8 must be the same as F_β for some β. Thus Lemma 9 is proved.

The next axiom to be discussed will be the Axiom of the Power Set. This is perhaps a more difficult axiom to verify. It will be an immediate consequence of the following.

<u>Lemma 10</u>: If $F_\beta(a) \subseteq F_\alpha(a)$, $\alpha \geq \omega$, then for some γ such that $\bar{\bar{\gamma}} \leq \bar{\bar{\alpha}}$, $F_\beta(a) = F_\gamma(a)$.

Here $\bar{\bar{\alpha}}$ denotes the cardinality of α. If δ is the first ordinal of cardinality greater than that of α, then by Lemma 8, if

$C = \{x \mid x \in T_\delta \ \& \ x \subseteq F_\alpha\}$, $C = F_\beta$ for some β and by Lemma 10, C is the power set of F_α in $\mathcal{N}$. This proves the Axiom of the Power Set. The proof of Lemma 10 will require certain auxiliary definitions. The basic idea is that since Lemma 10 holds for the constructible sets [3], the same proof can be carried out in the present situation taking into account the fact that all relations must eventually be forced.

Let A_1 and A_2 be two sets in $\mathcal{N}$, $A_1 \subseteq A_2$. Put $H(A_1) = \bigcup_{k=1}^{\infty} A_1^{(k)}$, where $A_1^{(k)}$ is the k-fold direct product of A_1 with itself. We should now like to introduce what may be called the "Skolem hull" of A_1 in A_2, i.e. a set containing A_1 such that those statements which are true in it are true A_2. We begin by enumerating all unlimited statements built up from the logical symbols, (containing no specific sets F_α in their construction) say $\mathfrak{A}_n(x; y_1, \ldots, y_{s_n})$. Define the function $x = L_n(y_1, \ldots, y_{s_n})$ for $y_i \in A_1$, if $x = F_\alpha$ and α is the least ordinal such that $F_\alpha \in A_2$ and $\tilde{\mathfrak{A}}_n(F_\alpha, y_1, \ldots y_{s_n})$ holds, where $\tilde{\mathfrak{A}}$ is the same as $\mathfrak{A}$ except that all quantifiers are restricted to A_1.** Let $\tilde{L}_n$ be an enumeration of all functions formed by iterated compositions of the L_n. Finally, let $Q(n,x,y)$ be the statement that $x \in H(A_1)$ and $y = \tilde{L}_n(x_1, \ldots)$.

<u>Lemma 11</u>: There is a single limited statement which is equivalent to $Q(n,x,y)$, and furthermore if $A_2 \subseteq T_\gamma$, the rank of $Q(n,x,y)$ is (α, r) for $\alpha \leq \gamma$.

Lemma 11 is a consequence of the same general principles previously given. The basic idea is that as long as we quantify over sets alone, definitions by means of transfinite induction can be given. Assume now that $F_\beta \subseteq F_\alpha$. Put $A_1 = T_\alpha \cup \{F_\beta\} \cup \{F_{\beta'}\}$ where $F_{\beta'} = \beta$, and $A_2 = T_\gamma$

**/see page 31.

where γ is the first ordinal greater than α, β and β'.

Definition 13.

For $\alpha' < \gamma$, define by transfinite induction $\Delta(\alpha')$ as follows

i) $\Delta(\emptyset) = \emptyset$

ii) $\Delta(\alpha') = \sup\{\Delta(\alpha'') \mid \alpha'' < \alpha'\}$, if for some P, $n \in \omega$ and $x \in H(A_1)$, P forces $Q(n,x,F_{\alpha'})$.

iii) $\Delta(\alpha') = \emptyset$, otherwise

Here sup means the first ordinal strictly greater than. Definition 13 is justified since $Q(n,x,y)$ is a relation in Z-F.

Lemma 12: For all $\alpha' < \gamma$, $\overline{\overline{\Delta(\alpha')}} \leq \bar{\bar{\alpha}}$.

Proof:

Since the cardinality of $H(A_1)$ is $\leq \bar{\bar{\alpha}}$ and there are only countably many P, the number of α' such that $\Delta(\alpha')$ is not zero is at most $\bar{\bar{\alpha}}$. By (ii) it follows that $\Delta(\alpha') \leq \bar{\bar{\alpha}}$.

In $\mathfrak{M}$ let A_3 be the set of all y such that $Q(n,x,y)$ holds for some $n \in \omega$, $x \in H(A_1)$. Clearly $A_3 \supseteq A_1$ and all statements formed from the sets in A_3 and the logical symbols which are true in A_2 are true in A_3. In particular, there must exist an ordinal in A_3 which constructs F_β from the set $\underline{a}$. (Here, we remark that saying a set s is an ordinal in A_3, merely means that $x \in s$, $y \in s \Longrightarrow$ either $x \in y$ or $y \in x$ and $x \in s$; $y \in x \Longrightarrow y \in s$, only if x and y are in A_3.) Let ψ be a mapping defined by means of transfinite induction for all x in A_3 as follows: $\psi(x) = \{z \mid \exists y \in A_3, y \in x, \psi(y) = z\}$. Then rank $\psi(F_{\alpha'}) \leq \Delta(\alpha')$. Also, ψ is an isomorphism with respect to $\in$ of A_3 onto $\psi(A_3)$, and is the identity on T_α

(since $x \in T_\alpha$, $y \in x \Longrightarrow y \in T_\alpha$) and on F_β (since $F_\beta \subseteq T_\alpha$) and hence $\psi(F_{\beta'})$ must construct F_β from $\underline{a}$ since $F_{\beta'}$ constructs F_β. But $\psi(F_{\beta'})$ is a genuine ordinal $\leq \bar{\bar{\alpha}}$, so that Lemma 10 is thus proved.

Finally, we note that all the other axioms of set theory are trivially true in $\mathcal{N}$ because of the construction of F_α and so we have completed the proof of Theorem 2.

§4. The Axiom of Choice

This section will be devoted to proving part 2 of Theorem 1. The basic definitions of forcing, limited statements etc. carry over with little change, except that we now introduce a group of symmetries in somewhat the same manner as one does when introducting individuals [4], [5], [6]. However, in our constructions the objects which satisfy certain symmetries will eventually become actual sets of integers which are distinct and so will not be truly symmetric. The important point is that some aspects of the symmetry will still be preserved.

Let a_{ij} be a collection of subsets of ω , for i and j integers, which are not necessarily in the model $\mathcal{M}$. We shall now define sets F_α in analogy with §2 , which are constructed from the a_{ij}. Since certain new problems arise we do not use the same notation as [3] but proceed slightly differently.

Put $U_i = \{a_{i1}, a_{i2}, \ldots \}$ and $V = \{ U_1, U_2, \ldots\}$. Our aim is to find a_{ij} such that if we put $\mathcal{N} = \{F_\alpha | \alpha \in \mathcal{M}\}$, the sets U_i will be disjoint and there will be no set in $\mathcal{N}$ which intersects each U_i in exactly one element. Thus to some extent we must treat the elements of any U_i , for fixed i , in a symmetrical manner when defining the sets F_α. This means that when we define the "collecting" process corresponding to $N(\alpha) = 0$ in §2, we must be certain that all the a_{ij} occur symmetrically in T_α.

Lemma 13: There exist unique functions $j(\cdot)$, $K_1(\cdot)$ and $K_2(\cdot)$ in $\mathcal{M}$ from ordinals to ordinals, and a function $N(\cdot)$ from ordinals to integers i, $0 \leq i \leq 8$ with the following properties:

1) $j(\alpha+1) > j(\alpha)$ and for all β such that $j(\alpha) < \beta < j(\alpha+1)$ the map $\beta \mapsto (N(\beta), K_1(\beta), K_2(\beta))$ is a 1-1 correspondence between all such β, and the set of all triples (i,γ,δ) where $1 \leq i \leq 8$, $\gamma \leq j(\alpha)$, $\delta \leq j(\alpha)$. Furthermore this map is order-preserving if the triples are given the order relation S [3].

2) $j(\emptyset) = \omega^2 + \omega + 1$ and if α is a limit ordinal $j(\alpha) = \sup\{j(\beta) | \beta < \alpha\}$.

3) for $\beta = j(\alpha)$ and for $\beta < \omega^2 + \omega + 1$, $N(\beta)$, $K_i(\beta)$ are put equal to zero.

The order relation S is introduced only to make the functions unique, though we never use this fact. The proof of Lemma 13 is a straightforward application of transfinite induction and the functions j, N, K_1 and K_2 are defined for all ordinals.

Definition 14.

Define $F_\alpha = \alpha$, if $\alpha \leq \omega$, F_α run through a_{ij} as $\omega < \alpha < \omega^2$; $F_{\omega^2} = U_1$, $F_{\omega^2+1} = U_2$, ... ; $F_{\omega^2+\omega} = V$; for $\alpha \geq \omega^2 + \omega + 1$,

if $N(\alpha) = 0$ put $F_\alpha = \{F_\beta | \beta < \alpha\}$

if $N(\alpha) = i > 0$, put $F_\alpha = \mathcal{F}_i(F_{K_1(\alpha)}, F_{K_2(\alpha)})$ where $\mathcal{F}_i$ are as in [3]. Put $\mathcal{N} = \{F_\alpha | \alpha \in \mathcal{M}\}$.

We now treat F_α as merely formal symbols and define limited and unlimited statements precisely as before except that we insist that in limited statements if we have $(x)_\alpha$ or $\exists_\alpha x$, then either $N(\alpha) = 0$ and $\alpha \geq \omega^2 + \omega + 1$ or $\alpha \leq \omega$. Also the definition of forcing is as before except that we now add that $U_i \in V$, $a_{ij} \in U_i$, and integers n belong or do not belong to a_{ij} if P contains precisely those conditions. Here P is a finite number of statements $n \in a_{ij}$ or $\neg n \in a_{ij}$ for finitely many n, i, j.

Let G denote the group of all permutations of a_{ij} which permute only finitely many a_{ij} in such a manner that the set U_i for all i is kept fixed (not necessarily elementwise). We extend the action of G to all F_α. If $g \in G$, $g(F_\alpha) = F_\alpha$ if $\alpha \leq \omega$, $g(U_i) = U_i$ and $g(V) = V$. If $\alpha \geq \omega^2 + \omega + 1$ and $N(\alpha) = 0$, $g(F_\alpha) = F_\alpha$. If $N(\alpha) > 0$ and if $J(\gamma) < \alpha < J(\gamma+1)$, $g(F_{K_1(\alpha)}) = F_{\alpha'}$, $g(F_{K_2(\alpha)}) = F_{\alpha''}$ then $g(F_\alpha) = F_\beta$ where β is the unique element in $J(\gamma) < \beta < J(\gamma+1)$ such that $N(\beta) = N(\alpha)$, $K_1(\beta) = \alpha'$, $K_2(\beta) = \alpha''$. This definition is justified since $K_1(\alpha) \leq J(\gamma)$, $K_2(\alpha) \leq J(\gamma)$ and by induction one can verify that $\beta < J(\gamma+1)$. Also we allow G to operate on conditions P and on statements $\mathfrak{A}$ in the obvious manner. If we have a limited statement involving $(x)_\alpha$ or $\exists_\alpha x$, when G operates these are left unchanged. This will cause no difficulties since we only allow α in $(x)_\alpha$ or $\exists_\alpha x$ if $N(\alpha) = 0$ and then $G(T_\alpha) = T_\alpha$. Finally, put G_m equal to the subgroup of G consisting of all $g \in G$ such that $g(a_{ij}) = a_{ij}$ if $i < m$.

Lemma 14: If P forces $\mathfrak{A}$, and $g \in G$, then $g(P)$ forces $g(\mathfrak{A})$.

Proof:

This is proved by induction on the rank of limited statements and

and going back to the definition of forcing. Again we remark that the invariance of T_α if $N(\alpha) = 0$ plays an important role.

Lemma 15: For each F_α, $\exists m$ such that G_m keeps F_α fixed.

Proof:

By induction. If $N(\alpha) = 0$, then G keeps it fixed. If $N(\alpha) > 0$, then if G_{m_1} keeps $F_{K(\alpha_1)}$ fixed, and G_{m_2} keeps $F_{K(\alpha_2)}$ fixed then G_{m_3} keeps F_α fixed, if $m_3 \geq m_1, m_2$.

List all ordinals α_n in $\mathcal{M}$ and all limited and unlimited statements $\mathcal{O}_n$ in $\mathcal{N}$. Let P_0 be empty and define P_{2n} to be the first extension of P_{2n-1} which forces either $\mathcal{O}_n$ or its negation. For odd indices we proceed differently. Let m be the smallest integer such that G_m leaves F_{α_n} invariant and such that for $i \geq m$, no a_{ij} occurs in P_{2n}. Let $\mathcal{O}$ be the statement $a_{m1} \in F_{\alpha_n}$. If P forces $\mathcal{O}$ then by Lemma 14, P also forces $a_{m2} \in F_{\alpha_n}$ and we define $P_{2n+1} = P_{2n}$. If no extension of P_{2n} forces $\mathcal{O}$, then no extension can force $a_{mj} \in F_{\alpha_n}$ for any j and again we define $P_{2n+1} = P_{2n}$. Assume that there is an extension P' of P_{2n} which forces $\mathcal{O}$. Let k be the first integer such that a_{mk} does not appear in P' and put P_{2n+1} equal to P_{2n} together with the same conditions on a_{mk} as P' has on a_{m1}. Then if g is the transposition (a_{m1}, a_{mk}), g leaves P_{2n+1} fixed. Since P_{2n+1} forces $\mathcal{O}$, it must also force $a_{mk} \in F_{\alpha_n}$.

Now the proof given in §2 can be repeated to show that if we define a_{ij} as the limit of the conditions P_n in the obvious way, and define $\mathcal{N} = \{F_\alpha | \alpha \in \mathcal{M}\}$, then $\mathcal{N}$ is a model for Z-F. Also, it is clear that the statement $a_{ij} = a_{k\ell}$ for $\langle i,j \rangle \neq \langle k,\ell \rangle$ can never be forced, since by extending any P it can be negated. Those statements in $\mathcal{N}$ which are true will be precisely those which are forced by some P_n and thus because of the definition of P_n, there will be no F_α such that F_α intersects each U_i in exactly one element. Thus in $\mathcal{N}$ the continuum has no well-ordering. This immediately implies that $\aleph_1 \neq 2^{\aleph_0}$ since $\aleph_1$ is well-ordered. If one restricts oneself to the collection of sets in $\mathcal{N}$ which are of finite type i.e. x is of type n if every element of x is of type $n-1$ and integers are of type zero, then one obtains an example of type theory where the Axiom of Choice fails.

§5. Relative Independence of the Continuum Hypothesis

This section is devoted to the proof of part 3 of Theorem 1. Let τ be any ordinal in $\mathcal{M}$, $\tau \geq 2$, and consider $\aleph_\tau$ which we recall [3] is the first ordinal of that cardinality. Let a_δ, for $\delta \in \aleph_\tau$, be a collection of subsets of ω not necessarily in $\mathcal{M}$. Put $U = \{a_\delta\}$ and $V = \{\langle a_\delta, a_{\delta'}\rangle \mid \delta < \delta'\}$. We define F_α as the sets constructed from the a_δ, U, and V. More precisely,

Definition 15.

1) $F_\alpha = \alpha$ if $\alpha \leq \omega$.
2) F_α enumerate a_δ as $\omega < \alpha < \aleph_\tau$.
3) $F_\alpha = U$ if $\alpha = \aleph_\tau$.
4) F_α enumerate $(a_\delta, a_{\delta'})$ for all δ, δ' in $\aleph_\tau$ as $\aleph_\tau < \alpha < 2\aleph_\tau$.
5) F_α enumerate $\langle a_\delta, a_{\delta'}\rangle$ for all δ, δ' in $\aleph_\tau$ as $2\aleph_\tau \leq \alpha < 3\aleph_\tau$.
6) $F_\alpha = V$ if $\alpha = 3\aleph_\tau$.
7) for $\alpha > 3\aleph_\tau$ define F_α as in §2, i.e. if $N(\alpha) = 0$, $F_\alpha = \{F_\beta \mid \beta < \alpha\}$ and if $N(\alpha) = i > 0$, then $F_\alpha = \mathcal{F}_i(F_{K_1(\alpha)}, F_{K_2(\alpha)})$.

Put $\mathcal{N} = \{F_\alpha | \alpha \in \mathcal{M}\}$ as before. In $\mathcal{N}$ it may happen that the meaning of certain cardinals may change. For example, it might happen that the a_δ are put into 1-1 correspondence with ω by some element of $\mathcal{N}$. Thus some care must be exercised when speaking of cardinals in $\mathcal{N}$. We repeat again the remark made earlier that for each α in $\mathcal{M}$, there is an α' in $\mathcal{M}$ such that $F_{\alpha'} = \alpha$ independently of the a_δ. This may be proved by observing that the rank of sets is increased "essentially" only when $N(\alpha) = 0$. Thus for each α, $\exists\alpha'$ such that rank $F_{\alpha'} = \alpha$ independently of a_δ. Then the set F_β which defines the rank of $F_{\alpha'}$ (if β is large enough) will necessarily be the ordinal α. It is also true that there is a function in $\mathcal{M}$, $\alpha' = f(\alpha)$ such that $F_{\alpha'} = \alpha$. We note that each F_α is a collection of F_β for some $\beta < \alpha$.

Now, we treat the F_α as formal objects as before. Conditions P are defined as before as finitely many conditions on the sets a_δ. Likewise, limited and unlimited statements, as well as the definition of forcing are precisely as before. We now define P_n by saying that P_o is empty while P_n is that extension of P_{n-1} which forces either the n^{th} statement or its negation. Let a_δ be the limit of P_n in the obvious sense, and $\mathcal{N} = \{F_\alpha\}$. We do not repeat the proof that $\mathcal{N}$ is a model for Z-F since the proof is precisely the same as in §2. It will be true that all a_δ will be distinct since the statement $a_\delta \neq a_{\delta'}$ if $\delta \neq \delta'$, is forced. This does not immediately imply that in the model $\mathcal{N}$ the power of the continuum is at least $\aleph_\tau$ since the meaning of $\aleph_\tau$ is not the same in $\mathcal{N}$ as in $\mathcal{M}$. We need some special definitions and lemmas to cover this situation.

Definition 16.

Let $\mathfrak{A}(x)$ be any fixed statement with one free variable. Let W be the set of all P such that for some α, P forces $\mathcal{O}(F_\alpha)$ and also forces the statement $\forall x(\mathcal{O}(x) \Longrightarrow x = F_\alpha)$. If P_1 and P_2 are in W we say $P_1 \sim P_2$ if they both force $\mathcal{O}(F_\alpha)$ for some α.

Remark that if $P \in W$, and P forces $\forall x(\mathcal{O}(x) \Longrightarrow x = F_\alpha)$ then if P forces $\mathcal{O}(F_\beta)$ then P also forces $F_\alpha = F_\beta$. For, if not, then for some $P' \supseteq P$, P' forces $F_\alpha \neq F_\beta$ and thus P' cannot force $\forall x(\mathcal{O}(x) \Longrightarrow x = F_\alpha)$ which is a contradiction. From this it follows that $\sim$ is an equivalence relation.

<u>Lemma 16</u>: The number of equivalence classes in W is countable.

Before proceeding to the proof, we repeat that as everywhere in the paper all set-theoretic notions unless specified otherwise refer to $\mathcal{M}$. Thus Lemma 16 means "countable in $\mathcal{M}$". Lemma 16 is rather surprising in view of the fact that there are uncountably many a_δ. The proof will only depend upon the fact that $P \in W$, $P' \supset P$ implies $P' \in W$ and $P' \sim P$, which follows immediately from the definition. Let P_1 be any element of W. Assume $P_1, P_2, \ldots, P_{n_k}$ have been chosen in W. Look at the set R_k of the finitely many conditions $m \in a_\delta$ or $\neg m \in a_\delta$ such that they or their negation belongs to some $P_1, \ldots P_{n_k}$. Let $P_{n_k+1}, \ldots, P_{n_{k+1}}$ be elements of W, such that if P' is any element in W, for some j, $n_k + 1 \leq j \leq n_{k+1}$, P' has precisely those conditions in R which P_j has. Since only finitely many possibilities occur, one can always find such P_j. Now, our contention is that for any $P \in W$, there is an n such that $P \sim P_n$. If P is not equivalent to P_1, then P must contain one of the conditions in R_1, since otherwise $P \cup P_1$ would be a permissible set of conditions (not containing any condition and its negation) and $P \sim P \cup P_1$, $P_1 \sim P \cup P_1$ so $P \sim P_1$.

Assume that P is equivalent to no P_n and that we have shown that P contains a condition in $R_{k'}$ which is not in $R_{k'-1}$ for $1 \leq k' \leq k$. Then for some j, $n_k < j \leq n_{k+1}$, P agrees with P_j with respect to the conditions in R_k. But since P is not equivalent to P_j, P must have a condition, whose negation is in P_j otherwise we can form $P \cup P_j$ and obtain $P \sim P \cup P_j \sim P_j$. Thus P has some condition in R_{k+1} which is not in R_k. Since P has only finitely many conditions this is a contradiction.

Let f be now a function in $\mathfrak{M}$, such that if $\alpha' = f(\alpha)$, $F_{\alpha'} = \alpha$ independently of a_δ. Let $\beta' = f(\beta)$.

<u>Lemma 17</u>: If $\bar{\bar{\alpha}} > \bar{\bar{\beta}}$ then in the model $\mathfrak{N}$ we will have $\bar{\bar{F}}_{\alpha'} > \bar{\bar{F}}_{\beta''}$.

<u>Proof</u>:

If the conclusion were false, then there would be a γ such that F_γ sets up a mapping from $F_{\beta'}$ onto $F_{\alpha'}$. Let

$$C = \{F_{\beta''} | \exists \beta_1 < \beta \;\&\; f(\beta_1) = \beta''\}, \quad D = \{F_{\alpha''} | \exists \alpha_1 < \alpha \;\&\; f(\alpha_1) = \alpha''\} .$$

Then F_γ maps C onto D. Now for each element in C, some P_n must force its image under F_γ to be some $F_{\alpha''}$ in D. For fixed x_o in C let $\mathfrak{O}(x)$ be the statement that x is the image of x_o under F_γ. Some P must then force the statement $\mathfrak{O}(F_{\alpha''}) \;\&\; \forall x (\mathfrak{O}(x) \Longrightarrow x = F_{\alpha''})$. Now it is clear that no two $F_{\alpha''}$ in D can ever be forced equal, so that by Lemma 16, for each x_o in C only countably many $F_{\alpha''}$ in D can ever be forced to be its image under F_γ. Now since $\bar{\bar{\beta}} > \bar{\bar{\alpha}} \cdot \aleph_o$, the range of F_γ cannot possibly be all of D and hence the lemma is proved.

Thus we can say that not only are ordinals in $\mathfrak{N}$ absolute (they always would be) but the cardinals are absolute as well.

Now in the model $\mathcal{N}$, the a_δ are of cardinality $\aleph_\tau$ hence we have that $2^{\aleph_0} \geq \aleph_\tau$.*** A repetition of the argument concerning the "Skolem hull" given in §3, (which we feel is a more elegant and conceptual version than the one given in [3]) will show that every subset of ω is equal to F_α where $\bar{\bar{\alpha}} \leq \aleph_\tau$. This shows that $\aleph_{\tau+1} \geq 2^{\aleph_0}$. We do not know what the true value of $2^{\aleph_0}$ is in the model $\mathcal{N}$, (perhaps this question itself is independent of Z-F set theory!) but it is well known that certain limit cardinals are excluded e.g. any cardinal which is a denumerable sum of smaller cardinals.

The Axiom of Choice is true since we have introduced the well-ordering V of the a_δ and hence it is easy to see that the construction of F_α may be described in $\mathcal{N}$, which in turn implies that $\mathcal{N}$ has a well-ordering given by $F_\alpha \prec F_\beta$ if $\alpha < \beta$. Thus part 3 of Theorem 1 is completely proved.

§6. Countable Axiom of Choice

The proof of part 4 of Theorem 1 parallels quite closely that of part 2. Let a^1_{ij} and a^2_{ij} be subsets of ω not necessarily in $\mathcal{M}$. Put

$$M^1_i = \{a^1_{ij}\}, \quad M^2_i = \{a^2_{ij}\}, \quad K_i = \{M^1_i, M^2_i\}, \quad L = \{K_1, K_2, \ldots\} .$$

We construct F_α as follows,

1) $F_\alpha = \alpha$ if $0 \leq \alpha \leq \omega$

2) F_α then successively enumerate a^1_{ij}, a^2_{ij}, M^1_i, M^2_i, K_i, L as $\omega < \alpha \leq 2\omega^2 + 3\omega$

3) if $\alpha > 2\omega^2 + 3\omega$, we define F_α precisely as in §4 except that now $2\omega^2 + 3\omega+1$ plays the role of $\omega^2 + \omega+1$

*** See page 31.

Let G denote the group of all permutations g of $a_{ij}^{(\cdot)}$ such that for all but finitely many i, g leaves $a_{ij}^{(\cdot)}$ fixed but for the other i maps a_{ij}^{1} into a_{ij}^{2} and conversely. We let G operate on all the F_α by defining g on M_i^1 and M_i^2 in the obvious way, g keeps K_i fixed and keeps L fixed. For $\alpha > 2\omega^2 + 3\omega$, g operates as in §4. Also the forcing definitions, limited and unlimited statements are as in §4. Let G_m be the set of all g which keep $a_{ij}^{(\cdot)}$ fixed if $i < m$.

Lemma 18: Each F_α is invariant under some G_n.

Proof:

As in §4.

We are now ready to define P_n. We let P_{2n} be the first extension of P_{2n-1} which forces the n^{th} statement. If α is the n^{th} ordinal in $\mathcal{M}$, assume F_α is invariant under G_m and that P_{2n} does not involve any $a_{ij}^{(\cdot)}$ with $i \geq m$. Consider the condition $M_m^1 \in F_\alpha$. If this condition is forced by P_{2n}, then clearly P_{2n} forces $M_m^2 \in F_\alpha$ since we have once more if P forces $\mathcal{O}\!\mathcal{L}$, $g(P)$ forces $g(\mathcal{O}\!\mathcal{L})$. In this case let $P_{2n+1} = P_{2n}$. If no extension $P' \supset P_{2n}$ is the first P' which forces $M_m^1 \in F_\alpha$, assume P' involves only $a_{mj}^{(\cdot)}$ with $j \leq j_0$. Let g be the permutation of order two such that

1) if $i = m$, $g(a_{ij}^{(\cdot)}) = a_{ij}^{(\cdot)}$,
2) $g(a_{mj}^{1}) = a_{m,j+j_0}^{2}$, $g(a_{mj}^{2}) = a_{m,j+j_0}^{1}$,
3) $g(a_{mj}^{1}) = a_{mj}^{2}$ if $i > 2j_0$.

Then there is an extension P'' of P' such that $g(P'') = P''$ and thus P'' forces $M_m^2 \in F_\alpha$. Put $P_{2n+1} = P''$. The resulting model $\mathcal{N} = \{F_\alpha \mid \alpha \in \mathcal{M}\}$. will not contain any choice function for the collection of pairs $\{M_i^1, M_i^2\}$.

One can further force the number of such pairs to be denumerable by introducing the set $\langle n, K_n \rangle$. Thus part 4 of Theorem 1 is proved.

§7. Methods of Proof

We should like to say something about the methods used. As the proof stands what we have shown is that if we adjoins to Z-F the axiom:

There is a set such that it forms a model for Z-F under ϵ

then one can deduce that there is a set which forms a model for Z-F and parts 1-4 of Theorem 1. However, it seems clear to us that a laborious rewriting of the proof would show that if we denote by Con(Z-F) and by Con(Z-F)_i , the number-theoretic statements which assert that no contradiction can be obtained in a finite number of steps stating with the axioms for Z-F or Z-F with parts 1-4 added respectively, then the statement $\text{Con(Z-F)} \Longrightarrow \text{Con(Z-F)}_i$ can be proved in elementary number theory. We merely attempt to sketch such a proof. Using Con(Z-F) one may assign a truth value to all statements such that the usual relationships hold. Call this method of evaluation E_1. By [3], since we know that $V = L$ is consistent, and since the arguments there given are also essentially arithmetical we may assume E_1 assigns truth to $V = L$. Now in any proof of a contradiction in (Z-F)_i define a new method of evaluation E_2 , by saying that the first statement is true if E_1 assigns truth to the statement that some P forces the same statement in $\mathfrak{N}$. By continuing this process with all the statements in the proof of contradiction, and by translating all the arguments given in this paper one shows that this leads to an absurdity since E_2 can be shown to be "coherent" in the obvious sense.

Our proof will also extend to give Theorem 1 even if we adjoin to Z-F axioms of infinity such as the existence of inaccessible cardinals. This is true for the same reason as Note 10 p. 69 [3], namely that these axioms imply the same assertions about the sets F_α. Again the only real difficulty arises in verifying the Axiom of the Power Set and the Axiom of Replacement.

Notes

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Here we need the fact that A_2 is a set such that if $F_\alpha \in A_2$, then $\exists \alpha'$ in A_2 and $\alpha = F_{\alpha'}$. This guarantees that we can speak about the <u>least</u> α in the definition of L_n. Any A_2 may be extended to an A_2 with this property. For, let f be a function in $\mathfrak{M}$ such that $F_{\alpha'} = \alpha$ if $\alpha' = f(\alpha)$, independently of a. For any A_2, put $B_n = B_{n-1} \cup \{F_{\alpha'} | \alpha' = f(\alpha) \ \& \ F_\alpha \in B_{n-1}\}$. Then the union of B_n satisfy this property.

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This is true since the well-ordering of the a_δ is in $\mathfrak{N}$ and is of the order type $\aleph_\tau$.

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Errata and Addenda to "Independence of ..."

Several people have pointed out some flaws in the presentation. In parts 2 and 4, the proof of the Power Set Axiom must be modified, since it is no longer true that the construction can be formalized in $\mathcal{N}$. In Gödel's original argument the isomorphism ψ on p. 19 was not defined that way, but was defined as an isomorphism of the indices α which preserved the relations K_1, K_2, J_i, etc. Then he showed that this isomorphism preserved the ϵ relation between the F_α. I attempted to give a short-cut by directly establishing the second property, noting that since the construction can be formalized, the indices take care of themselves. The best way to proceed is as follows:

Let $C(P,\alpha)$ denote the least β such that P forces $F_\beta \epsilon F_\alpha$, 0 if none exists. If we wish to show that $F_\beta \subseteq F_\alpha \Longrightarrow F_\beta = F_{\beta'}$ with $\bar{\bar{\beta}}' \leq \bar{\bar{\alpha}}$, let S denote the smallest collection of ordinals containing all ordinals $\leq \alpha$, the ordinal β, closed under K_1, K_2, J_i (as in Gödel) and $C(P,\alpha)$ for all P. One easily shows (in parts 2 and 4) $\bar{\bar{S}} = \bar{\bar{\alpha}}$. Let g be an isomorphism with respect to ϵ, K_1, K_2, J_i of S onto a set of ordinals all of cardinality $\leq \bar{\bar{\alpha}}$. (Actually, this is an initial segment as in Gödel and g need only be an isomorphism with respect to ϵ, the others are automatic). In the model $\mathcal{N}$, S will be closed with respect to $C(\alpha) =$ least $\beta \ni F_\beta \epsilon F_\alpha$, since every statement is forced. Also $F_{g(\alpha)} \epsilon F_{g(\beta)} \Longleftrightarrow F_\alpha \epsilon F_\beta$ so that $F_\beta = F_{g(\beta)}$, as desired. This method also avoids any Skolem-Löwenheim construction.

In Lemma 16, in general $\approx$ is not an equivalence relation, although in the application it is. The best way to state Lemma 16 is, any collection of mutually incompatible P is countable.

A detailed examination of the proof of the Power Set Axiom yields the result that if $\aleph_\tau$ is not $\leq$ a sum of Countable ~~fewer cardinals~~ of smaller cardinality, then the continuum is of cardinality $\aleph_\tau$. Thus we will have $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$, if $\tau = 2$. One can also show that we may have models in which only countably many reals are constructible.

The last section devoted to the question of formalizing the proof is rather vague. The simplest way, as was pointed out by several people, is to show that the consistency proof for a finite set of the axioms for $(Z\text{-}F)_1$ requires using only finitely many axioms of Z-F. The existence of a countable model satisfying these can be proved in Z-F. The argument I gave is not model-theoretic. It is based on the observation that given any finite number of statements, one can decide whether they will be true or false in $\mathcal{N}$ by using in place of $\mathcal{M}$, the original universe of sets. This is because the only use of the countability of $\mathcal{M}$ is in evaluating all possible statements. Of course, the order in which the statements are considered will affect the choice of P_1, P_2 etc. The two ideas are really very similar, the first probably being more familiar.