The Schwarzschild Metric and Applications

Analytic solutions of Einstein's equations are hard to come by. It's easier in situations that exhibit symmetries.

1916: Karl Schwarzschild sought the metric describing the static, spherically symmetric spacetime surrounding a spherically symmetric mass distribution.

A static spacetime is one for which there exists a time coordinate t such that

i) all the components of $g_{\mu\nu}$ are independent of t

ii) the line element $d\mathbf{s}^2$ is invariant under the transformation $t \rightarrow -t$

A spacetime that satisfies (*i*) but not (*ii*) is called stationary. An example is a rotating azimuthally symmetric mass distribution.

The metric for a static spacetime has the form $d\mathbf{s}^2 = A(x^i)\, c^2\, dt^2 - d\mathbf{l}^2$

where x^i are the spatial coordinates and $d\mathbf{l}^2$ is a time-independent spatial metric. Cross-terms $dt\, dx^i$ are missing because their presence would violate condition (*ii*).

[Note: The Kerr metric, which describes the spacetime outside a rotating axisymmetric mass distribution, contains a term $\propto dt\, d\phi$.]

To preserve spherical symmetry, $d\mathbf{l}^2$ can be distorted from the flat-space metric only in the radial direction. In flat space, (1) r is the distance from the origin and (2) $4\pi r^2$ is the area of a sphere. Let's define r such that (2) remains true but (1) can be violated. Then,

$$d\mathbf{l}^2 = B(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

$A(x^i) \rightarrow A(r)$ in cases of spherical symmetry.

$$\Rightarrow \qquad d\mathbf{s}^2 = A(r)\, c^2\, dt^2 - B(r)\, dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

The Ricci tensor for this metric is diagonal, with components

$$\frac{R_{tt}}{c^2} = -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB} \quad , \qquad R_{rr} = \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB}$$

$$R_{\theta\theta} = -1 + \frac{1}{B} + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) \quad , \qquad R_{\phi\phi} = \sin^2\theta\, R_{\theta\theta}$$

SP 10.1

Primes denote differentiation with respect to r.

The region outside the spherically symmetric mass distribution is empty.

The vacuum Einstein equations are $R_{\mu\nu} = 0$. To find $A(r)$ and $B(r)$:

1. $$\frac{B}{A}\frac{R_{tt}}{c^2} + R_{rr} = 0 \quad \Rightarrow \quad -\frac{A'}{rA} - \frac{B'}{rB} = 0 \quad \Rightarrow \quad A'\,B + A\,B' = 0$$

$$\Rightarrow \quad (AB)' = 0 \quad \Rightarrow \quad AB = \text{const} \equiv \alpha \quad \Rightarrow \quad B = \frac{\alpha}{A}$$

2. Substitute this result for B in the expression for $R_{\theta\theta}$, set it equal to zero, and solve for A:

$$-1 + \frac{A}{\alpha} + \frac{rA}{2\alpha}\left(\frac{A'}{A} + \frac{A'}{A}\right) = 0 \quad \Rightarrow \quad -\alpha + A + r\,A' = 0 \quad \Rightarrow \quad A + r\,A' = \alpha$$

$$\frac{d(rA)}{dr} = \alpha \quad \Rightarrow \quad rA = \alpha\,(r + k)$$ (k is another constant of integration)

Thus, $$A = \alpha\left(1 + \frac{k}{r}\right) \quad , \quad B = \left(1 + \frac{k}{r}\right)^{-1}$$

Note: We only used the sum of R_{tt} and R_{rr} to solve for A and B. We must also verify that R_{tt} and R_{rr} vanish individually (exercise for home).

To find α and k, consider large r, where the weak field limit applies.
In this limit, coordinate $r \to$ physical r and

$$g_{tt} = 1 + \frac{2\phi}{c^2} = 1 - \frac{2GM}{c^2 r}$$ (Topic 8, p. 30)

$$\Rightarrow \quad \alpha\left(1 + \frac{k}{r}\right) = 1 - \frac{2GM}{c^2 r} \quad \Rightarrow \quad \alpha = 1 \;,\; k = -\frac{2GM}{c^2}$$

Thus, the Schwarzschild metric is

$$d\mathbf{s}^2 = \left(1 - \frac{a}{r}\right) c^2\, dt^2 - \left(1 - \frac{a}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

with $a = \dfrac{2GM}{c^2}$ ("Schwarzschild radius")

Birkhoff's Theorem: The Schwarzschild metric describes any spherically symmetric spacetime outside the mass/energy distribution, even if the distribution moves (in a spherically symmetric way).

This implies that a radially pulsating spherically symmetric star does not produce gravitational radiation.

As noted on p. 2, if t and r are constant, then the metric describes the surface of a sphere, with polar angle θ and azimuthal angle ϕ, and surface area $4\pi r^2$.

Thus, r is known as the "area distance".

As $r \to \infty$, the metric becomes Minkowskian ("asymptotic flatness")

Measuring distances and times:

1. The metric blows up at $r = a$ => we need different coords to describe the region $r \leq a$, if this region is empty.

   $a = 2.9$ km for the Sun's mass and $a = 0.88$ cm for the Earth's mass.

2. The spatial and temporal parts of the metric are separate. For static spacetimes, such coords can always be chosen.

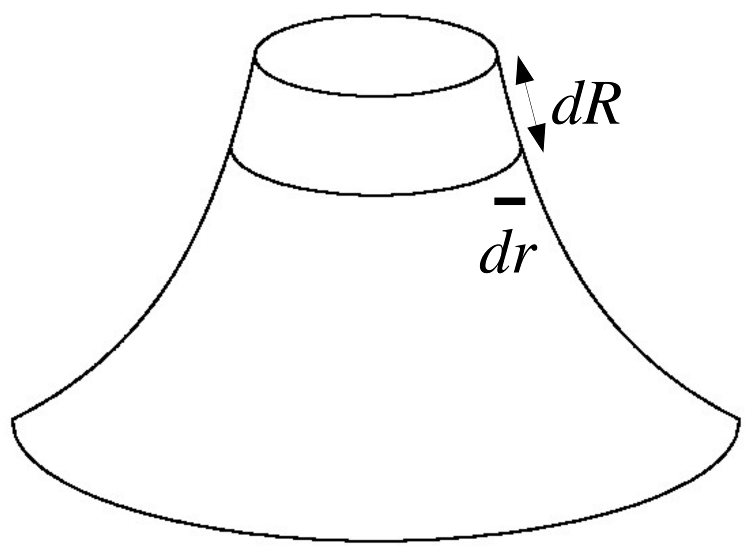
$$ds^2 = \left(1 - \frac{a}{r}\right) c^2\, dt^2 - \left(1 - \frac{a}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta^2\, d\phi^2\right)$$

3. When $a = 0$, the metric is flat. When $a > 0$, both the spatial and temporal parts are curved.

4. When $a > 0$, the coord distance (i.e., area dist), r, does not measure the radial distance R btwn 2 points with the same θ, ϕ:

$$dR = (1 - a/r)^{-1/2}\, dr > dr$$

Lampshade analogy:



Distance from one circle to the next is dR, but increase in circumference radius is dr.

The center of the circle (or sphere) doesn't even have to be in the space!

5. The proper time elapsed on a clock at a fixed position in space is:

$$d\tau = (1 - a/r)^{1/2}\, dt$$

SP 10.2

Stationary clocks do not tick coordinate time!

(as we saw in our earlier thought experiment)

Gravitational frequency shift:

Suppose a light signal is sent from (r_E, θ_E, ϕ_E) to (r_R, θ_R, ϕ_R)

Coordinate time of emission = t_E , reception = t_R

$ds^2 = 0$ => $g_{44}\, c^2\, dt^2 = -g_{ij}\, dx^i\, dx^j$

With u an affine parameter for the null geodesic of the signal:

$$\frac{dt}{du} = \frac{1}{c}\left[\left(1 - \frac{a}{r}\right)^{-1} (-g_{ij})\, \frac{dx^i}{du}\, \frac{dx^j}{du}\right]^{1/2}$$

$$t_R - t_E = \frac{1}{c}\int_{u_E}^{u_R}\left[\left(1-\frac{a}{r}\right)^{-1}(-g_{ij})\,\frac{dx^i}{du}\,\frac{dx^j}{du}\right]^{1/2} du$$

RHS depends only on the path through space, so $t_R - t_E$ is the same for different signals: $t_{R,1} - t_{E,1} = t_{R,2} - t_{E,2}$

=> $t_{R,2} - t_{R,1} = t_{E,2} - t_{E,1}$, or, $\Delta t_R = \Delta t_E$

Clock records proper time:

$$\Delta\tau_E = (1 - a/r_E)^{1/2}\,\Delta t_E$$

$$\Delta\tau_R = (1 - a/r_R)^{1/2}\,\Delta t_R$$

=> $$\frac{\nu_R}{\nu_E} = \frac{\Delta\tau_E}{\Delta\tau_R} = \left(\frac{1 - a/r_E}{1 - a/r_R}\right)^{1/2}$$

$$\frac{\nu_R}{\nu_E} = \frac{\Delta\tau_E}{\Delta\tau_R} = \left(\frac{1 - a/r_E}{1 - a/r_R}\right)^{1/2}$$

If $r_E \gg a$ and $r_R \gg a$, then:

$$\frac{\nu_R}{\nu_E} \approx \left(1 - \frac{1}{2}\frac{a}{r_E}\right)\left(1 + \frac{1}{2}\frac{a}{r_R}\right) \approx 1 + \frac{a}{2}\left(\frac{1}{r_R} - \frac{1}{r_E}\right)$$

$$= 1 + \frac{GM}{c^2}\left(\frac{1}{r_R} - \frac{1}{r_E}\right) = 1 - \frac{\Delta\phi}{c^2} = 1 - \frac{gh}{c^2}$$

Last expression: when light rises height *h* near Earth's surface; same result we found earlier using the photon concept

Grav redshift has been confirmed to first order in *a/r*, which is a test of the EP. Need to go to higher order to test Schwarzschild metric; not yet accomplished.

Solar System Tests of GR

1. Precession of Mercury's perihelion

Start with the Newtonian theory of orbits:

Specific angular momentum, $\mathbf{h} = \mathbf{r} \times \mathbf{v}$

$$\frac{d\mathbf{h}}{dt} = \mathbf{r} \times \frac{\mathbf{F}}{m} = \mathbf{r} \times \left(-\frac{GM}{r^2}\right)\hat{r} = 0$$

$\mathbf{r} \times \mathbf{v}$ = const => particle moves in a single plane ($\perp$ $\mathbf{h}$); we'll assume $\theta = \pi/2$

Total specific energy (kinetic plus potential) is $E = \frac{1}{2}v^2 - \frac{GM}{r}$

With $\mathbf{v} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}$, $E = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\phi}^2 - \frac{GM}{r}$

$$E = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\phi}^2 - \frac{GM}{r} \quad ; \quad h = |\mathbf{h}| = |\mathbf{r} \times (\dot{r}\hat{r} + r\dot{\phi}\hat{\phi})| = r^2\dot{\phi}$$

Define $u = 1/r$: $\dot{r} = -\dot{u}u^{-2}$

Energy equation becomes: $\dot{u}^2u^{-4} + u^{-2}h^2u^4 - 2GMu = 2E$

Substituting from the angular momentum equation:

$$h^2\left(\frac{du}{d\phi}\right)^2 + h^2u^2 = 2GMu + 2E$$

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{2GMu}{h^2} + \frac{2E}{h^2}$$

Differentiate: $2\frac{du}{d\phi}\frac{d^2u}{d\phi^2} + 2\frac{du}{d\phi}u = \frac{2GM}{h^2}\frac{du}{d\phi}$

=> $$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2}$$

Solution: $$u = \frac{GM}{h^2}\left[1 + e\cos\left(\phi - \phi_0\right)\right]$$

=> $$r = \frac{h^2/GM}{1 + e\cos(\phi - \phi_0)}$$

Circle if $e = 0$, ellipse if $0 \le e < 1$ (as for planets orbiting the Sun)

Perihelion (closest approach to Sun) occurs when $\phi = \phi_0$ (maximizes denominator in expression for r)

Relativistic theory:

$$L = g_{\mu\nu}\,\dot{x}^\mu\,\dot{x}^\nu \;\; = \left(1-\frac{a}{r}\right)c^2\dot{t}^2 - \left(1-\frac{a}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\,\dot{\phi}^2$$

Euler-Lagrange equation for θ :

$$\frac{d}{du}\left(-2r^2\dot{\theta}\right) + 2r^2\sin\theta\cos\theta\,\dot{\phi}^2 = 0$$

$$r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} - r^2\sin\theta\cos\theta\,\dot{\phi}^2 = 0$$

If initially $\theta = \pi/2$ and $\dot{\theta} = 0$, then $\ddot{\theta} = 0$

and the particle stays in the plane $\theta = \pi/2$

With $\theta = \pi/2$: $$L = \left(1 - \frac{a}{r}\right) c^2 \dot{t}^2 - \left(1 - \frac{a}{r}\right)^{-1} \dot{r}^2 - r^2 \dot{\phi}^2$$

t and ϕ are ignorable coords, so:

$$\frac{\partial L}{\partial \dot{t}} = \text{const} \qquad \frac{\partial L}{\partial \dot{\phi}} = \text{const}$$

=> $$\left(1 - \frac{a}{r}\right) \dot{t} = k \qquad r^2 \dot{\phi} = h$$

Adopt τ as the parameter and divide the metric equation by $d\tau^2$:

$$\left(1 - \frac{a}{r}\right) c^2 \dot{t}^2 - \left(1 - \frac{a}{r}\right)^{-1} \dot{r}^2 - r^2 \dot{\phi}^2 = c^2$$

$$\left(1 - \frac{a}{r}\right)^2 c^2 \dot{t}^2 - \dot{r}^2 = \left(1 - \frac{a}{r}\right) \left(c^2 + r^2 \dot{\phi}^2\right)$$

$$c^2 k^2 - \dot{r}^2 = \left(1 - \frac{a}{r}\right) \left(c^2 + \frac{h^2}{r^2}\right)$$

$$c^2k^2 - \dot{r}^2 = \left(1 - \frac{a}{r}\right)\left(c^2 + \frac{h^2}{r^2}\right)$$

Again, define $u = 1/r$:

$$c^2k^2 - u^{-4}\dot{u}^2 = (1 - au)(c^2 + h^2u^2)$$

$$c^2k^2 - h^2\left(\frac{du}{d\phi}\right)^2 = c^2 - ac^2u + h^2u^2 - ah^2u^3$$

Differentiate:

$$-2h^2\frac{du}{d\phi}\frac{d^2u}{d\phi^2} = -ac^2\frac{du}{d\phi} + 2h^2u\frac{du}{d\phi} - 3ah^2u^2\frac{du}{d\phi}$$

$$\frac{d^2u}{d\phi^2} + u = \frac{ac^2}{2h^2} + \frac{3}{2}au^2$$

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2} + \frac{3GM}{c^2}u^2$$

Ratio of final term to the previous term is: $\frac{3u^2h^2}{c^2} = \frac{3v^2r^2}{r^2c^2} = 3\frac{v^2}{c^2} \ll 1$

For Mercury, $3v^2/c^2 \approx 10^{-7}$

=> the last term is a small correction; u does not deviate greatly from its classical value

Simple approximation: replace u^2 in the final term with the classical solution u_0

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2} + \frac{3GM}{c^2}\frac{G^2M^2}{h^4}[1 + e\cos(\phi - \phi_0)]^2$$

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2} + \frac{3(GM)^3}{c^2h^4}\left[1 + 2e\cos(\phi - \phi_0) + e^2\cos^2(\phi - \phi_0)\right]$$

Equation for a harmonic oscillator with a constant force and two oscillatory forcing terms.

Of the three terms in the brackets:

1st : a small correction to the size of the orbit (a small change in the constant forcing)

3rd : imposes small oscillations on the orbit

Both of the above effects are too small to observe.

2nd term: leads to a precession of the perihelion; too small to observe on a single orbit, but it accumulates over successive orbits, making it observable over long times.

Ignoring the 1st and 3rd terms in brackets:

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2} + \frac{6e(GM)^3}{c^2h^4}\cos(\phi - \phi_0)$$

Solution: $$u = u_0 + \frac{3e(GM)^3}{c^2h^4}\phi\sin(\phi - \phi_0)$$

$$u = \frac{GM}{h^2}\left[1 + e\cos(\phi - \phi_0) + 3e\left(\frac{GM}{ch}\right)^2 \phi\sin(\phi - \phi_0)\right]$$

$$= \frac{GM}{h^2}\left\{1 + e\cos\left[\phi - \phi_0 - 3\left(\frac{GM}{ch}\right)^2 \phi\right]\right\}$$

Last step took $\sin x \approx x$ and $\cos x \approx 1$. $x = 3\left(\frac{GM}{ch}\right)^2\phi$; $\frac{GM}{ch} \ll 1$

Result is a precessing ellipse.

$$u = \frac{GM}{h^2}\left\{1 + e\cos\left[\phi - \phi_0 - 3\left(\frac{GM}{ch}\right)^2 \phi\right]\right\}$$

The argument of the cosine changes by 2π when ϕ changes by:

$$\Delta\phi = 2\pi\left[1 - 3\left(\frac{GM}{ch}\right)^2\right]^{-1} \approx 2\pi\left[1 + 3\left(\frac{GM}{ch}\right)^2\right]$$

Thus, perihelion precesses by angle $\psi \approx 6\pi\left(\frac{GM}{ch}\right)^2$ per orbit.

The perihelion and aphelion distances are $r_{p/a} = \frac{h^2/GM}{1 \pm e}$

The semimajor axis is $a_s = \frac{1}{2}(r_p + r_a) = \frac{h^2}{GM(1 - e^2)}$

=> $$\psi = \frac{6\pi GM}{c^2}\left(\frac{GM}{h^2}\right) = \frac{6\pi GM}{c^2(1-e^2)a_s}$$

Sun's mass $M = 2 \times 10^{33}$ g $G = 6.67 \times 10^{-8}$ (cgs)

Mercury: $a_s = 0.39$ AU $= 5.8 \times 10^{12}$ cm , $e = 0.206$

orbital period = 0.24 yrs

=> $$\psi = 5 \times 10^{-7}\,\text{rad} = 0.104''$$ => 43" per century

Observed perihelion shift is 574" per century. Almost of this is due to perturbations by the other planets. The remainder is 43" per century, in exact agreement with GR.

2. Deflection of Light Grazing the Sun

Lagrangian is the same as for massive particles, but we must adopt a parameter other than τ. As for massive particles:

$$\left(1-\frac{a}{r}\right)\dot{t}=k_p \qquad r^2\dot{\phi}=h_p$$

(Dot denotes differentiation wrt the affine parameter and subscript “p” denotes the photon case.)

$ds^2 = 0$ => $$\left(1-\frac{a}{r}\right)c^2\dot{t}^2-\left(1-\frac{a}{r}\right)^{-1}\dot{r}^2-r^2\dot{\phi}^2=0$$

$$\dot{r}^2=\left(1-\frac{a}{r}\right)^2c^2\dot{t}^2-r^2\dot{\phi}^2\left(1-\frac{a}{r}\right)$$

$$\dot{r}^2=c^2k_p^2-\left(1-\frac{a}{r}\right)\frac{h_p^2}{r^2}$$

$$\dot{r}^2 = c^2 k_p^2 - \left(1 - \frac{a}{r}\right) \frac{h_p^2}{r^2} \quad \Rightarrow \quad u^{-4} \dot{u}^2 = c^2 k_p^2 - (1 - au) h_p^2 u^2$$

$$\Rightarrow \quad h_p^2 \left(\frac{du}{d\phi}\right)^2 = c^2 k_p^2 - (1 - au) h_p^2 u^2$$

$$2 h_p^2 \frac{du}{d\phi} \frac{d^2 u}{d\phi^2} = -2 h_p^2 u \frac{du}{d\phi} + 3 h_p^2 a u^2 \frac{du}{d\phi}$$

$$\frac{d^2 u}{d\phi^2} + u = \frac{3}{2} a u^2$$

(RHS $\ll u$ since $a \ll r$ => $au \ll 1$)

In absence of RHS,
solution is a straight line:

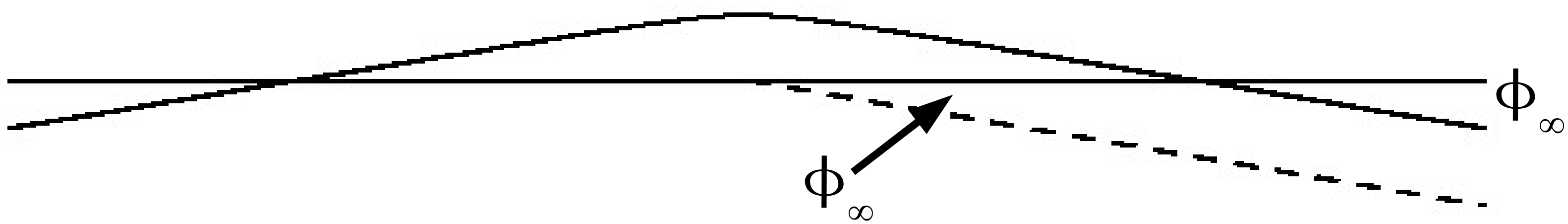
$$u = \frac{\sin\phi}{R} \quad ; \quad r \sin\phi = R$$

Use first approx for u in RHS:

$$\frac{d^2 u}{d\phi^2} + u = \frac{3 a \sin^2\phi}{2 R^2}$$

$$\frac{d^2u}{d\phi^2} + u = \frac{3a\sin^2\phi}{2R^2}$$

Solution: $$u = \frac{\sin\phi}{R} + \frac{3a}{4R^2}\left(1 + \frac{1}{3}\cos 2\phi\right)$$



As $u \to 0 \ (r \to \infty), \ \phi \to \phi_\infty$

$$\phi_\infty \ll 1 \quad \Rightarrow \quad \frac{\phi_\infty}{R} = -\frac{3a}{4R^2}\left(1 + \frac{1}{3}\right) = -\frac{a}{R^2} \quad \Rightarrow \quad \phi_\infty = -\frac{a}{R}$$

By symmetry, total deflection is $\xi = |2\phi_\infty| = \frac{2a}{R} = \frac{4GM}{c^2 R}$

Radius of Sun = 6.96×10^{10} cm

=> $\xi = 8.5 \times 10^{-6}$ rad = 1.75''

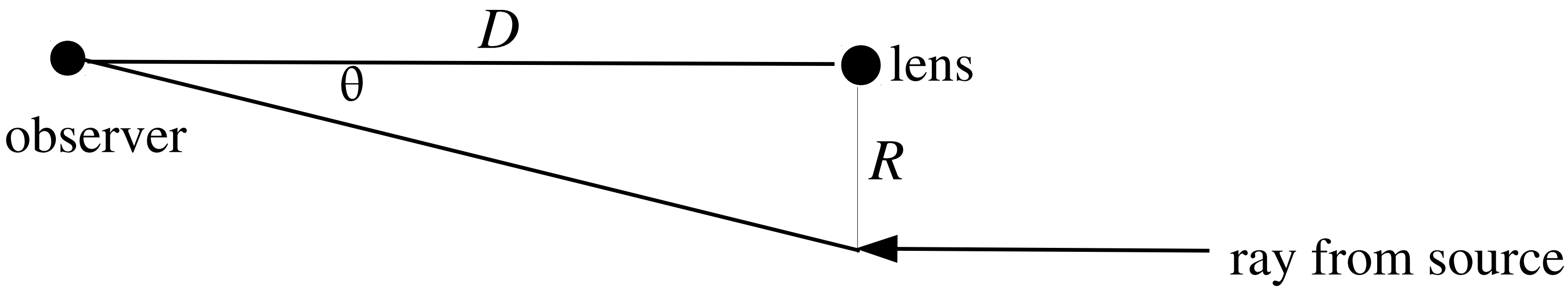
First verified (to within ≈20%) by Eddington in 1919.

Optical: hard to improve precision much

Radio: look at quasars; accuracy of 10^{-4}

Gravitational lensing is now widely used in astronomy.
(See Bartelmann 2010, Classical and Quantum Gravity, 27, 233001 for a review.)

In the simplest geometry, a point source of radiation lies directly behind a spherically symmetric massive body.



A gravitational lens with mass *M* is located a distance *D* from the observer.
A ray located a perpendicular distance *R* from the center of the lens is bent through angle $\theta = \frac{4GM}{c^2 R} \approx \frac{R}{D}$

If the lens radius < *R*, then the observer sees a luminous ring, called an "Einstein ring". In terms of *M* and *D*,

$$R = \frac{\sqrt{4GMD}}{c}$$

=> the angular radius of the ring $\theta_E = \frac{R}{D} = \frac{1}{c}\sqrt{\frac{4GM}{D}}$ ("Einstein radius")

The impact parameter of the ray is also called the Einstein radius: $R_E = \frac{\sqrt{4GMD}}{c}$

Very few Einstein rings have been observed, because it is highly unlikely that the source will lie directly behind the lens and that the lens will be spherically symmetric.

In other geometries, lensing may still be observed on angular scales $\sim\theta_E$: ring arcs; multiple, distorted images of the source; magnification. Classified as "strong lensing" when there are multiple images and "weak lensing" when there is just a distortion of the one image.

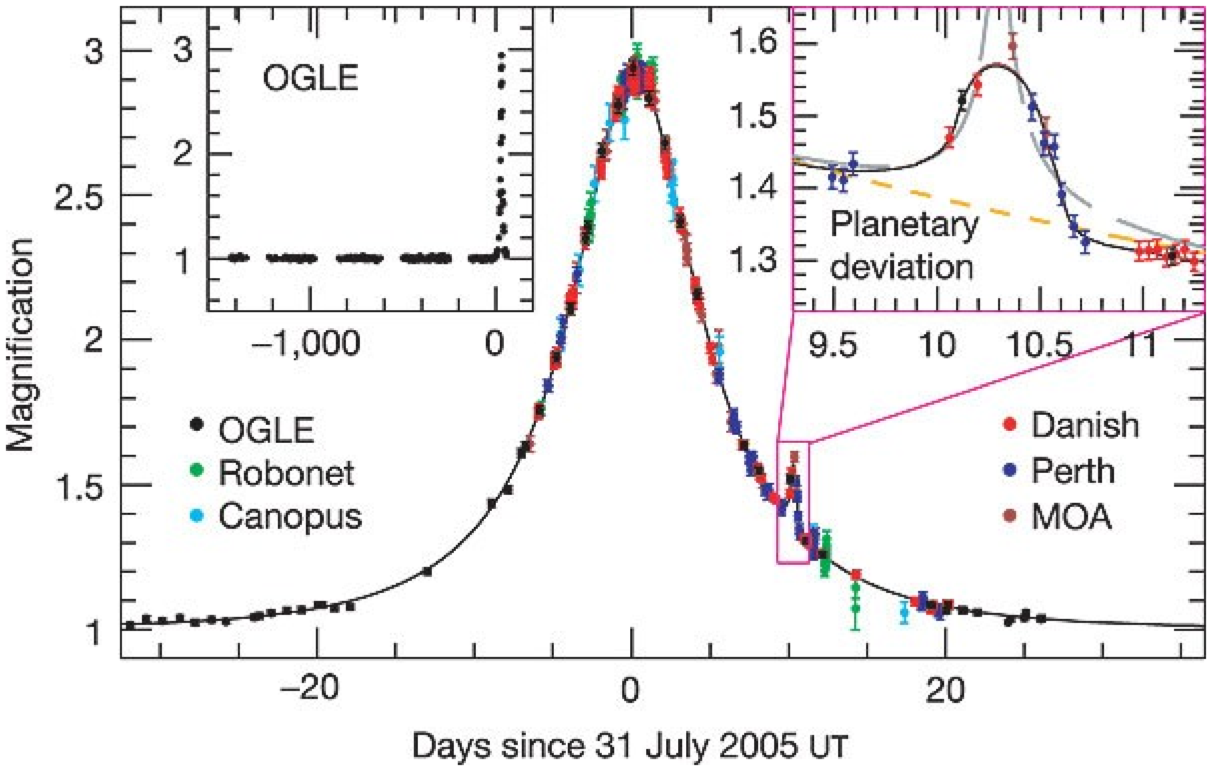
The larger θ_E, the more likely there are to be background sources that will be lensed. Probability is $\sim 10^{-6}$ when the lens is a star in the Galaxy. Probability is higher for distant galaxies and ~ 1 for galaxy clusters observed with today's powerful telescopes (increase in M more than compensates for increase in D).

Lensing by stars in the Galaxy is called microlensing

$\theta_E \ll 1''$ => lensing images are not resolved. Instead, magnification as the lens star passes in front of the source star produces a characteristic light curve, with a timescale ~ weeks to months.

MACHO microlensing survey (Alcock et al. 2000, Astrophysical Journal, 542, 281) sought to infer the mass density of compact objects (low-mass stars, stellar remnants) in the Galaxy. It observed 11.9 million stars over 5.7 years and found 13-17 events. They concluded that 8% to 50% of the Galaxy's dark matter halo could be in the form of MACHOs (massive compact halo objects). So, MACHOs cannot account for all of the dark matter.
The more recent EROS-2 survey (Tisserand et al. 2007, A&A, 469, 387) concludes that <8% of the dark matter halo could be in the form of MACHOs.

If a star has a planet located at a distance $\approx R_E$, then the planet can produce a detectable lensing signature.



(5.5 Earth mass planet at 2.6 AU separation; Beaulieu et al. 2006, Nature, 439, 437)

To date, 133 microlensing planets have been discovered in 118 systems. (Jan 14, 2021)

(See the Extrasolar Planets Encyclopaedia at http://exoplanet.eu/catalog.php)

Lensing by galaxies and clusters of galaxies have multiple applications, including:

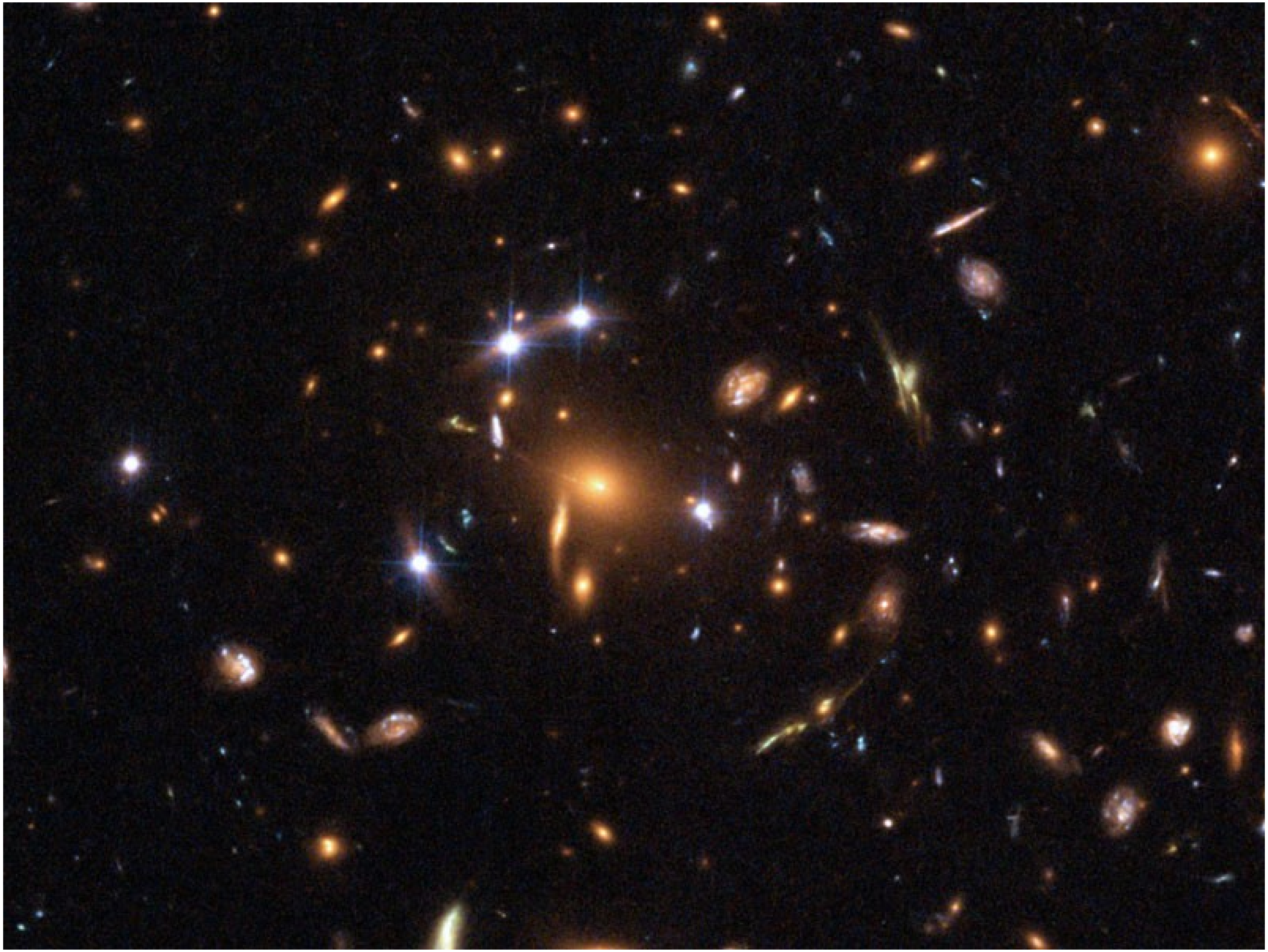
1. Probing the mass distribution in the galaxy or cluster

2. Estimating Hubble's constant, if the source luminosity is time variable and multiple images are observed

3. Use gravitational lensing as a telescope to get better info about the source

Lensing examples on the following pages are from Astronomy Picture of the Day (http://antwrp.gsfc.nasa.gov/apod/).

Einstein Cross: The foreground lensing galaxy has redshift z = 0.0394 while the source (4 images) is a quasar at redshift z = 1.695.



Multiple images of a quasar and other sources; lens is cluster SDSS J1004+4112

Lensing by cluster Abell 1689; purple is modeled dark matter distribution

3. Shapiro Delay

Shapiro (1964, Phys Rev Lett, 13, 789) noted that when light grazes the Sun, the observed travel time is longer than it would be for Newtonian gravity. This is not simply due to the increased length of the deflected path.

Shapiro and collaborators sent radar beams to the inner planets near superior conjunction (planet across the Sun from Earth) and measured the time to receive the reflected signal. The additional time predicted by GR (beyond the Newtonian result), $\Delta t_{\text{excess}} \sim 100$'s of μs. This corresponds to 10's of km at the speed of light.

Numerous complications plague the experiment:

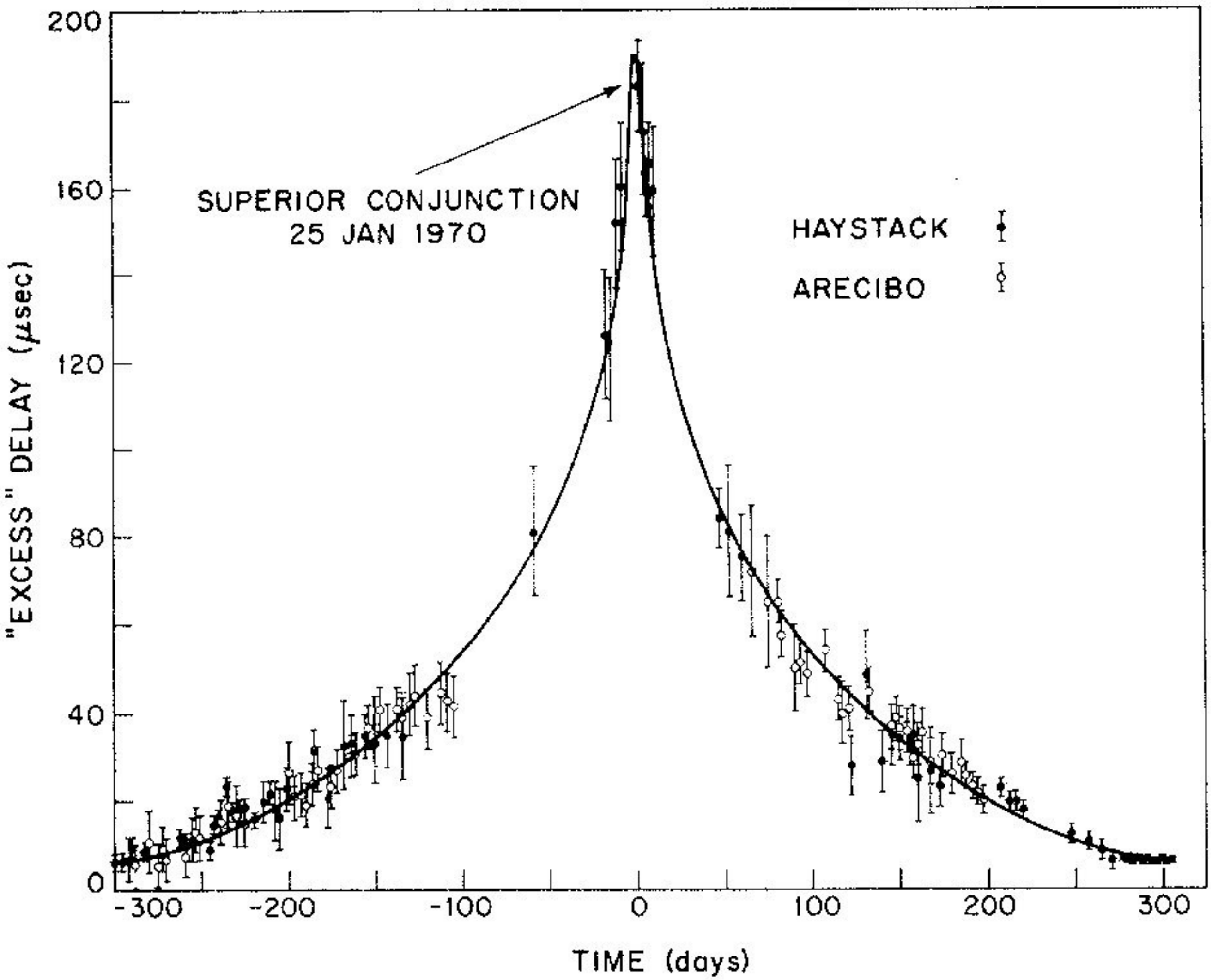
1. The expected delay time (in both Newtonian theory and GR) depends on the distances of the planets from the Sun (R_p and R_E) and the distance of closest approach, R (approximately the Sun's radius). These are not known to sufficient precision. Furthermore, there is not a unique radial coordinate in the Schwarzschild geometry. We made one convenient choice, but others are possible. Which of these should we identify with R_p, R_E, R?

2. The radar beam is refracted when it passes through the solar corona. This has to be modeled. (Use multiple radar frequencies.)

3. Planet surfaces are rough. So, there is a dispersion of delay time comparable to Δt_{excess} itself. (Use satellites orbiting the planets with transponders that introduce a frequency shift in the radar beam, separating its signal from the planetary reflection. Other complicated techniques can deal with the dispersion without the use of satellites.)

4. The motion of the Earth and planet need to be taken into account, as well as effects from their own gravitational fields.

Strategy has been to measure delay times at numerous times (not just superior conjunction), producing a plot of excess radar delay time vs. time (i.e., vs. orientation of Earth-Sun-planet). This curve can be extremely well fit using GR prediction and appropriate choices of R_p, R_E, R.



SP 10.3

Shapiro et al. (1971, Phys Rev Lett, 26, 1132) Earth-Venus result, verifying GR prediction to within 5%. Later experiments improved precision to 0.1% (Reasenberg et al. 1979, Astrophysical Journal, 234, L219; Viking at Mars), 0.002% (Bertotti et al. 2003, Nature, 425, 374; Cassini spacecraft).

Black Holes

Occur if the radius of the central mass < the Schwarzschild radius, a

$$ds^2 = \left(1 - \frac{a}{r}\right) c^2\, dt^2 - \left(1 - \frac{a}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

As $r \to a,\ g_{rr} \to \infty$: What's going on?

To find out, let's examine the behavior of particles on radial trajectories.

From the metric: $$c^2 = \left(1 - \frac{a}{r}\right) c^2 \dot{t}^2 - \left(1 - \frac{a}{r}\right)^{-1} \dot{r}^2$$

Recall (p. 14): $$\left(1 - \frac{a}{r}\right) \dot{t} = k$$

=> $$c^2 \left(1 - \frac{a}{r}\right) = c^2 k^2 - \dot{r}^2$$

$$\dot{r}^2 - c^2 k^2 + c^2 \left(1 - \frac{a}{r}\right) = 0$$

$$\dot{r}^2 - c^2k^2 + c^2\left(1 - \frac{a}{r}\right) = 0$$

Recall (p. 7): $d\tau = \left(1 - \frac{a}{r}\right)^{1/2} dt$

Suppose particle is at rest when $r = r_0$ => $k^2 = 1 - a/r_0$

=> $$\dot{r}^2 - c^2\left(1 - \frac{a}{r_0} - 1 + \frac{a}{r}\right) = 0$$

$$\dot{r}^2 = ac^2\left(\frac{1}{r} - \frac{1}{r_0}\right) \quad \Rightarrow \quad \frac{1}{2}\dot{r}^2 = GM\left(\frac{1}{r} - \frac{1}{r_0}\right)$$

Exactly the same form as conservation of energy in Newtonian theory,
but: *i*) the dot denotes diff wrt τ, not t *ii*) r is area (not radial) distance

$$\frac{d\tau}{dr} = -(2GM)^{-1/2}\left(\frac{r_0 r}{r_0 - r}\right)^{1/2}$$ as the particle falls radially inward.

Integrating the previous equation:

$$\Delta\tau = -(2GM)^{-1/2}\int_{r_0}^{r}\left(\frac{r_0 r}{r_0 - r}\right)^{1/2} dr$$

Clearly, $r = a$ is reached in finite proper time τ.

Also: scalars constructed from the Riemann tensor (e.g. $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$) are finite at $r = a$; the particle can pass $r = a$ without a catastrophe.

How much coordinate time elapses during the fall?

$$\frac{dt}{dr} = \frac{dt}{d\tau}\frac{d\tau}{dr} = k\left(1 - \frac{a}{r}\right)^{-1}\frac{d\tau}{dr}$$

$$= -(2GM)^{-1/2}\left(1 - \frac{a}{r_0}\right)^{1/2}\left(1 - \frac{a}{r}\right)^{-1}\left(\frac{r_0 r}{r_0 - r}\right)^{1/2}$$

$$= -\left(\frac{r_0 - a}{2GM}\right)^{1/2}\frac{r^{3/2}}{(r-a)(r_0 - r)^{1/2}}$$

Coordinate time to fall from $r = r_0$ to $r = a + \epsilon$ is:

$$t_\epsilon = -\left(\frac{r_0 - a}{2GM}\right)^{1/2} \int_{r_0}^{a+\epsilon} \frac{r^{3/2}dr}{(r-a)(r_0 - r)^{1/2}}$$

Given that $a + \epsilon < r$: $r > a$ and $r_0 - r < r_0$

$$\Rightarrow \quad t_\epsilon > \left[\frac{(r_0 - a)a^3}{2GMr_0}\right]^{1/2} \int_{a+\epsilon}^{r_0} \frac{dr}{r-a}$$

$$= \left[\frac{(r_0 - a)a^2}{r_0 c^2}\right]^{1/2} \ln\left(\frac{r_0 - a}{\epsilon}\right) \quad \rightarrow \infty \quad \text{as} \quad \epsilon \rightarrow 0$$

How does the fall look to a stationary observer at $r > a$?

Need to consider the paths of photons leaving the falling particle and arriving at the observer, i.e., radial null geodesics.

$$d\mathbf{s}^2 = 0 \;\Rightarrow\; \left(1 - \frac{a}{r}\right) c^2 \dot{t}^2 = \left(1 - \frac{a}{r}\right)^{-1} \dot{r}^2$$

$$\Rightarrow \quad \frac{dr}{dt} = c\left(1 - \frac{a}{r}\right) \quad \to 0 \quad \text{as} \;\; r \to a$$

=> any photon that the observer sees must have been emitted when the particle was still at $r > a$

Outward-traveling light just sits still at $r = a$ (not even any angular motion). Also, the frequency observed at $r > a$ is zero; infinite redshift

$$ds^2 = \left(1 - \frac{a}{r}\right) c^2\, dt^2 - \left(1 - \frac{a}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

The signs of the dt^2 and dr^2 terms change when r crosses a;
r becomes the time coordinate! There's only one direction to time;
for $r < a$, that's the direction of decreasing r.

=> Any particle at $r < a$ must fall inwards. Same holds for photons
(since they must travel forward in time as seen by a particle)

=> nothing can escape from $r < a$.

$r = a$ is known as the "event horizon".

At $r = 0$, there is a true singularity, with scalars constructed from the Riemann tensor $\rightarrow \infty$.