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## On $r$ -reflexive Banach spaces

IRYNA BANAKH, TARAS BANAKH, ELENA RISS

*Abstract.* A Banach space  $X$  is called  $r$ -reflexive if for any cover  $\mathcal{U}$  of  $X$  by weakly open sets there is a finite subfamily  $\mathcal{V} \subset \mathcal{U}$  covering some ball of radius 1 centered at a point  $x$  with  $\|x\| \leq r$ . We prove that an infinite-dimensional separable Banach space  $X$  is  $\infty$ -reflexive ( $r$ -reflexive for some  $r \in \mathbb{N}$ ) if and only if each  $\varepsilon$ -net for  $X$  has an accumulation point (resp., contains a non-trivial convergent sequence) in the weak topology of  $X$ . We show that the quasireflexive James space  $J$  is  $r$ -reflexive for no  $r \in \mathbb{N}$ . We do not know if each  $\infty$ -reflexive Banach space is reflexive, but we prove that each separable  $\infty$ -reflexive Banach space  $X$  has Asplund dual. As a by-product of the proof we obtain a covering characterization of the Asplund property of Banach spaces.

*Keywords:* reflexive Banach space,  $r$ -reflexive Banach space, Asplund Banach space

*Classification:* 46A25, 46B10, 46B22

### 1. Introduction

In this paper we address the following problem posed by the third author in 2000 at the Winter School in Křišťanovice (Czech Republic):

**Question 1.** *Is a separable Banach space  $X$  reflexive if each net in  $X$  has an accumulation point in the weak topology of  $X$ ?*

By a *net* in a Banach space  $(X, \|\cdot\|)$  we understand an  $\varepsilon$ -net  $N \subset X$  for some  $\varepsilon > 0$ . A subset  $N \subset X$  is called an  $\varepsilon$ -net for a subset  $B \subset X$  if for every point  $x \in B$  there is a point  $y \in N$  with  $\|x - y\| < \varepsilon$ .

It turns out that Question 1 is equivalent to an even more intriguing question concerning  $\infty$ -reflexive Banach spaces.

**Definition 1.** A Banach space  $(X, \|\cdot\|)$  is called  $r$ -reflexive where  $r \in [0, +\infty]$  if for every cover  $\mathcal{U}$  of  $X$  by weakly open sets there is a finite subfamily  $\mathcal{V} \subset \mathcal{U}$  that covers the open unit ball  $x + B_X = \{y \in X : \|x - y\| < 1\}$  centered at some point  $x \in X$  with  $\|x\| \leq r$ .

Observe that a Banach space  $X$  is reflexive if and only if it is 0-reflexive. We define a Banach space  $X$  to be  $\omega$ -reflexive if it is  $r$ -reflexive for some  $r \in [0, \infty)$ .

It turns out that for infinite-dimensional separable Banach spaces the property appearing in Question 1 is equivalent to the  $\infty$ -reflexivity.

**Theorem 1.** *An infinite-dimensional separable Banach space  $X$  is  $\infty$ -reflexive (resp.  $\omega$ -reflexive) if and only if every net in  $X$  has an accumulation point (resp. contains a non-trivial convergent sequence) in the weak topology of  $X$ .*

This theorem is not true for non-separable Banach spaces: for any uncountable set  $\Gamma$  the Banach space  $c_0(\Gamma)$  is weakly Lindelöf [Fab, §7.1]. Consequently, each net for  $c_0(\Gamma)$ , being uncountable, has an accumulation point in the weak topology. On the other hand, the space  $c_0(\Gamma)$  is not  $\infty$ -reflexive by Proposition 2 below. This example also shows that in the realm of non-separable Banach spaces the answer to Question 1 is negative.

Theorem 1 allows us to reformulate and extend Question 1 as follows:

**Question 2.** *Is a (separable) Banach space  $X$  reflexive if it is  $\infty$ -reflexive?  $\omega$ -reflexive? 1-reflexive?*

In light of the last part of this question, it is interesting to mention that a Banach space  $X$  is reflexive if and only if  $X$  is  $r$ -reflexive for some  $r < 1$ . This equivalence (observed by the referee) follows from the fact that each cover of a 1-ball  $x + B_X$  centered at a point  $x \in X$  with  $\|x\| \leq r < 1$  covers also the closed ball of radius  $\frac{1}{2}(1 - r)$  centered at the origin.

Trying to answer Questions 1 and 2, it is natural to look at the quasireflexive James space  $J$  (having codimension 1 in its second dual). We recall that a Banach space  $X$  is *quasireflexive* if it has finite codimension in its second dual space  $X^{**}$ .

**Theorem 2.** *The quasireflexive James space  $J$  is not  $\omega$ -reflexive.*

However we do not know if the James space is  $\infty$ -reflexive.

**Question 3.** *Is each quasireflexive Banach space  $\infty$ -reflexive? Is the James space  $\infty$ -reflexive?*

Our principal result on separable  $\infty$ -reflexive Banach spaces asserts that any such a space has Asplund dual. We recall that a Banach space  $X$  is *Asplund* if each separable subspace  $Y$  of  $X$  has separable dual  $Y^*$ .

**Theorem 3.** *Each separable  $\infty$ -reflexive Banach space  $X$  has Asplund dual  $X^*$ .*

Since the Banach space  $l_1$  is not Asplund, Theorem 3 implies the result of [Ba] (asserting that the dual space  $X^*$  of a separable  $\infty$ -reflexive Banach space  $X$  contains no copy of  $l_1$ ). Theorem 3 has also another corollary related to the Fréchet-Urysohn property of the weak topology on bounded subsets of an  $\infty$ -reflexive Banach space.

Following [En, §1.6], we say that a topological space  $X$  is *Fréchet-Urysohn* if for each accumulation point  $x \in X$  of a subset  $A \subset X$  some sequence  $\{a_n\}_{n=1}^\infty \subset A$  converges to  $x$ .

Since Eberlein compact spaces are Fréchet-Urysohn, the weak topology of a reflexive space  $X$  is Fréchet-Urysohn on bounded subsets of  $X$ . A similar property holds for separable  $\infty$ -reflexive Banach spaces.

**Corollary 1.** *If  $X$  is a separable  $\infty$ -reflexive Banach space, then the unit ball of  $X$  endowed with the weak topology is a Fréchet-Urysohn space.*

PROOF: First we show that the space  $X$  contains no copy of the Banach space  $l_1$ . In the opposite case the non-Asplund space  $l_\infty = l_1^*$  would be a quotient of the Asplund space  $X^*$  which is not possible (because the Asplund property is preserved by quotients). Since  $X$  contains no copy of  $l_1$ , it is legal to apply the Odell-Rosenthal Theorem [OR] (see also [Dis, p. 215]) to conclude that the second dual unit ball  $\bar{B}^{**}$  endowed with the weak\* topology is Rosenthal compact; more precisely,  $\bar{B}^{**}$  is a compact subspace of the space  $B_1(B^*) \subset \mathbb{R}^{\bar{B}^*}$  of functions of the first Baire class on the dual unit ball  $\bar{B}^*$ . Finally, we apply the Bourgain-Fremelin-Talagrand Theorem [BFT] establishing the Fréchet-Urysohn property of separable Rosenthal compacta to conclude that the unit ball  $\bar{B}^{**} \supset \bar{B}$  is Fréchet-Urysohn.  $\square$

The proof of Theorem 3 relies on a characterization of the Asplund property of a dual Banach space in terms of so-called weak\* covering properties.

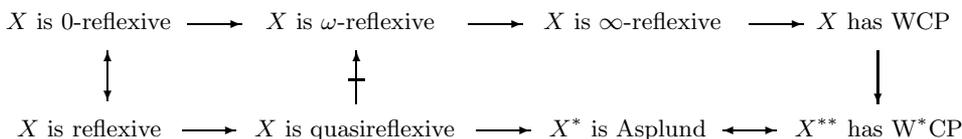
**Definition 2.** A Banach space  $X$  is said to satisfy the  $\tau$ -covering property, where  $\tau$  is a weaker linear topology on  $X$ , if for every sequence  $(U_i)_{i=1}^\infty$  of  $\tau$ -open sets in  $X$  whose intersection  $\bigcap_{i=1}^\infty U_i$  is a norm-neighborhood of the origin in  $X$  there are points  $x_1, \dots, x_n \in X$  such that the union  $\bigcup_{i=1}^n (x_i + U_i)$  contains the open unit ball  $B_X$  centered at the origin of  $X$ .

If  $\tau$  is the weak or weak\* topology, then we call *the weak or weak\* covering properties*, briefly, WCP and W\*CP.

Theorem 3 can be derived from the following theorem that can have an independent value.

- Theorem 4.** (1) *Each separable  $\infty$ -reflexive Banach space has the weak covering property.*  
 (2) *If a Banach space  $X$  has the weak covering property, then the second dual space  $X^{**}$  has the weak\* covering property.*  
 (3) *A Banach space  $X$  is Asplund if and only if the dual space  $X^*$  has the weak\* covering property.*

The obtained results fit into the following diagram connecting various reflexivity-like properties and holding for any separable Banach space  $X$ :



Before passing to proofs of Theorems 1–4 we discuss some stability properties of  $r$ -reflexive spaces and ask some related questions.

**Proposition 1.** *Let  $Z$  be a Banach subspace of a Banach space  $X$ .*

- (1) *If  $X$  is  $r$ -reflexive for some  $r \in [0, +\infty]$ , then the quotient space  $X/Z$  is  $r$ -reflexive too.*
- (2) *If  $X$  is  $r$ -reflexive for some  $r \in \{0, \omega, \infty\}$ , then each Banach space  $Y$  isomorphic to  $X$  is  $r$ -reflexive.*
- (3) *If  $Z$  is reflexive and  $X/Z$  is  $r$ -reflexive for some  $r \in [0, +\infty)$ , then  $X$  is  $r$ -reflexive too.*

**Question 4.** *Is a subspace of a (separable)  $r$ -reflexive Banach space  $r$ -reflexive (at least for  $r \in \{\omega, \infty\}$ )?*

**Question 5.** *Is the second dual  $X^{**}$  of an  $r$ -reflexive Banach space  $X$   $r$ -reflexive? Is a Banach space  $X$   $r$ -reflexive if its second dual  $X^{**}$  is  $r$ -reflexive for some  $r \in [0, +\infty]$ ?*

Since the  $r$ -reflexivity is an isomorphic property for  $r \in \{0, \omega, \infty\}$ , we may also ask:

**Question 6.** *Is the  $r$ -reflexivity an isomorphic property for arbitrary  $r \in (0, +\infty)$ ?*

As we already know, Theorem 1 is not true for non-separable Banach spaces. What about Theorem 3?

**Question 7.** *Has each  $\infty$ -reflexive Banach space Asplund dual?*

We can give a partial answer for Banach spaces with  $\aleph_0$ -monolithic dual space. We recall that a topological space  $X$  is *monolithic* (resp.  *$\aleph_0$ -monolithic*) if each (separable) subspace  $Y$  of  $X$  has network weight  $\text{nw}(Y)$  equal to the density  $\text{dens}(Y)$  of  $Y$ . It is easy to see that each Banach space is monolithic in norm and weak topologies.

We shall say that a Banach space  $X$  has ( $\aleph_0$ -) *monolithic dual space*, if the dual space  $X^*$  is ( $\aleph_0$ -)monolithic with respect to the weak\* topology. It can be shown that a Banach space  $X$  has ( $\aleph_0$ -)monolithic dual space if and only if for any (separable) subset  $Y \subset X^*$  the annihilator  $Y^\top = \{x \in X : \forall y^* \in Y \ y^*(x) = 0\}$  satisfies  $\text{dens}(X/Y^\top) = \text{dens}(Y)$  in  $X$ . The latter property was introduced in [BPZ] as the property (1). Since Corson compacta are monolithic, each weakly Lindelöf determined Banach space (=Banach space with Corson dual ball) has monolithic dual. In particular, for each set  $\Gamma$  the Banach space  $c_0(\Gamma)$  has monolithic dual.

**Proposition 2.** *Each  $\infty$ -reflexive Banach space with  $\aleph_0$ -monolithic dual has Asplund dual.*

PROOF: Assume that  $X$  is an  $\infty$ -reflexive Banach space with  $\aleph_0$ -monolithic dual. To show that  $X^*$  is Asplund, take any separable subspace  $Y \subset X^*$  and consider its annihilator  $Y^\top = \{x \in X : \forall y^* \in Y \ y^*(x) = 0\}$  in  $X$ . The Hahn-Banach Theorem implies that  $Y$  is weak\* dense in  $(X/Y^\top)^*$  identified with the subspace  $(Y^\top)^\perp \subset X^*$  of functionals that annihilate  $Y^\top$ . Since  $X$  has  $\aleph_0$ -monolithic dual, the space  $(X/Y^\top)^*$ , being separable, has countable network weight in the weak\* topology. Consequently, the unit ball of  $(X/Y^\top)^*$  in the weak\* topology has countable network weight and is metrizable. This is equivalent to the separability of  $X/Y^\top$ . Being a quotient of the  $\infty$ -reflexive space  $X$ , the space  $X/Y^\top$  is  $\infty$ -reflexive. Applying Theorem 3, to the separable  $\infty$ -reflexive space  $X/Y^\top$ , we conclude that the dual space  $(X/Y^\top)^*$  is Asplund and consequently, its separable subspace  $Y$  has separable dual  $Y^*$ .  $\square$

Also we do not know if the separability assumption is essential in Corollary 1.

**Question 8.** *Let  $X$  be an  $\infty$ -reflexive Banach space (with  $\aleph_0$ -monolithic dual). Is the unit ball of  $X$  endowed with the weak topology a Fréchet-Urysohn space?*

Finally, we ask:

**Question 9.** *Let  $X$  be a separable  $\infty$ -reflexive Banach space. Is the dual space  $X^*$  separable? Equivalently, is the second dual  $X^{**}$  separable?*

Now we present the proofs of the results announced in the introduction.

## 2. Proof of Theorem 4

The first item of Theorem 4 is established in

**Lemma 1.** *A separable  $\infty$ -reflexive Banach space  $X$  has the weak covering property.*

PROOF: To show that  $X$  has the weak covering property, take any sequence  $(U_n)_{n \in \omega}$  of weakly open sets in  $X$  such that  $\bigcap_{n \in \omega} U_n$  has non-empty interior in  $X$ . Let  $\{x_n : n \in \omega\}$  be a countable dense set in  $X$ . It follows that  $\{x_n + U_n : n \in \omega\}$  is a cover of  $X$  by weakly open sets. The  $\infty$ -reflexivity of  $X$  yields a point  $x \in X$  such that the open unit ball  $x + B_X$  centered at  $x$  lies in the finite union  $\bigcup_{n=0}^m x_n + U_n$  for some  $m \in \omega$ . Then  $B_X \subset \bigcup_{n=0}^m (x_n - x + U_n)$  witnessing the weak covering property of  $X$ .  $\square$

The second item of Theorem 4 is established in

**Lemma 2.** *If a Banach space has the weak covering property, then the second dual space  $X^{**}$  has the weak\* covering property.*

PROOF: Suppose that  $(V_i)_{i=1}^\infty$  is a sequence of weak\* open sets in  $X^{**}$  whose intersection  $\bigcap_{i=1}^\infty V_i$  contains a closed  $\varepsilon$ -ball  $\varepsilon \bar{B}^{**}$ . To show that  $X^{**}$  has the

weak\* covering property, it suffices to find points  $x_1, \dots, x_n \in X^{**}$  such that  $\bigcup_{i=1}^n (x_i + V_i) \supset \bar{B}^{**}$ .

By the compactness of  $\varepsilon \bar{B}^{**}$  and the regularity of the weak\* topology, for every  $i \in \mathbb{N}$ , there is a weak\* open subset  $W_i \subset X^{**}$  such that  $\varepsilon \bar{B}^{**} \subset W_i \subset \overline{W_i} \subset V_i$  where the closure is taken in the weak\* topology of  $X^{**}$ .

Consider the sequence  $(U_i)_{i=1}^\infty$ ,  $U_i = W_i \cap X$ , of weakly open sets in  $X$ . Note that  $\bigcap_{i=1}^\infty U_i = (\bigcap_{i=1}^\infty W_i) \cap X \supset \varepsilon \bar{B}^{**} \cap X = \varepsilon \bar{B}$ . By definition of the weak covering property of  $X$ , there exist points  $x_1, \dots, x_n \in X$  such that the union  $\bigcup_{i=1}^n (x_i + U_i)$  contains the open unit ball  $B$  centered at the origin. According to Goldstine Theorem [HHZ, p.46],  $\bar{B} = \bar{B}^{**}$ . Thus we obtain  $\bar{B}^{**} = \bar{B} \subset \bigcup_{i=1}^n (x_i + \overline{U_i}) \subset \bigcup_{i=1}^n (x_i + \overline{W_i}) \subset \bigcup_{i=1}^n (x_i + V_i)$ , and hence  $X^{**}$  has the weak\* covering property.  $\square$

For the proof of the third item of Theorem 4 we need an auxiliary

**Lemma 3.** *Let  $K$  be a weak\* compact subset of a weak\* open set  $U$  of a dual Banach space  $X^*$ . Then there is a weak\* open set  $V$  in  $X^*$  such that  $K \subset V \subset U$  and  $V = V + L$  for some weak\* closed linear subspace  $L$  of finite codimension in  $X^*$ .*

PROOF: By definition, the weak\* topology on  $X^*$  has a base consisting of sets  $W$  such that  $W = W + F^\perp$  for some finite subset  $F \subset X$ . Here, as expected,  $F^\perp = \{x^* \in X^* : \forall x \in F \ x^*(x) = 0\}$ . Consequently, for every  $x \in K$  we may find a weak\* open subset  $O(x) \subset X^*$  such that  $x \in O(x) \subset U$  and  $O(x) = O(x) + F_x^\perp$  for some finite subset  $F_x \subset X$ . Using the weak\* compactness of  $K$ , choose a finite subcover  $\{O(x_1), \dots, O(x_n)\}$  of the cover  $\{O(x) : x \in K\}$  of  $K$ . Then the weak\* open set  $V = \bigcup_{i=1}^n O(x_i)$  has the properties  $K \subset V \subset U$  and  $V = V + F^\perp$ , where  $F = \bigcup_{i=1}^n F_{x_i}$ .  $\square$

The following characterization establishes the third item of Theorem 4.

**Proposition 3.** *For a Banach space  $X$ , the following conditions are equivalent:*

- (1)  $X$  is Asplund;
- (2)  $X^*$  has  $W^*CP$ ;
- (3) for each sequence  $(U_i)_{i=1}^\infty$  of weak\* open subsets of  $X^*$  whose intersection  $\bigcap_{i=1}^\infty U_i$  is a norm-neighborhood of the origin there is a sequence of points  $\{a_i^*\}_{i=1}^\infty \subset X^*$  such that  $X^* = \bigcup_{i=1}^\infty (a_i^* + U_i)$ .

PROOF: (1)  $\Rightarrow$  (3) Fix a sequence  $(U_i)_{i=1}^\infty$  of weak\* open sets in  $X^*$  whose intersection  $\bigcap_{i=1}^\infty U_i$  contains the closed  $\varepsilon$ -ball  $\varepsilon \bar{B}^*$  centered at the origin.

By Lemma 3, for every  $i \geq 1$  there exists a weak\* open set  $V_i \subset X^*$  such that  $\varepsilon \bar{B}^* \subset V_i \subset U_i$  and  $V_i = V_i + F_i^\perp$  for some finite subset  $F_i \subset X$ . Let  $Y$  be the closed linear hull of the set  $F = \bigcup_{i=1}^\infty F_i$  in  $X$ . As  $X$  is Asplund,  $Y^*$  is separable. Since  $Y^*$  is isomorphic to  $X^*/Y^\perp = X^*/F^\perp$ , the latter quotient space is separable. Since the quotient map  $\pi : X^* \rightarrow X^*/F^\perp$  is open, the set  $\pi(\varepsilon \bar{B}^*)$

has non-empty interior in  $X^*/F^\perp$ . The separability of  $X^*/F^\perp$  yields a countable subset  $C = \{c_i : i \geq 1\}$  of  $X^*/F^\perp$  such that  $C + \pi(\varepsilon B^*) = X^*/F^\perp$ . For every  $i \geq 1$  find a point  $a_i^* \in X^*$  with  $\pi(a_i^*) = c_i$ . Then

$$\begin{aligned} X^* &= \bigcup_{i=1}^{\infty} (a_i^* + \pi^{-1}(\pi(\varepsilon B^*))) \\ &= \bigcup_{i=1}^{\infty} (a_i^* + \varepsilon B^* + F^\perp) \subset \bigcup_{i=1}^{\infty} (a_i^* + (V_i + F_i^\perp)) \subset \bigcup_{i=1}^{\infty} (a_i^* + U_i). \end{aligned}$$

The implication (3)  $\Rightarrow$  (2) trivially follows from the weak\* compactness of the unit ball  $\bar{B}^* \subset X^*$ .

(2)  $\Rightarrow$  (1) Assume that  $X$  is not Asplund. Then by Theorem 5.2.3 of [Fab], the dual Banach space  $X^*$  contains a bounded subset  $D$  such that every non-empty relatively weak\* open subset  $U$  of  $D$  has norm diameter  $> 8\varepsilon$  for some  $\varepsilon > 0$ . Without loss of generality,  $0 \in D$  and  $\|x^*\| < 1$  for every  $x^* \in D$ .

Let  $2 = \{0, 1\}$  and  $2^{<\omega} = \bigcup_{n \in \omega} 2^n$  be the set of all finite binary sequences. For each sequence  $s = (s_0, \dots, s_{n-1}) \in 2^{<\omega}$ , by  $|s| = n$  we denote its length and by  $s|k = (s_0, \dots, s_{k-1})$  the initial segment of  $s$  of length  $k \leq |s|$ . For  $i \in \{0, 1\}$  let  $s\hat{i} = (s_0, \dots, s_{n-1}, i)$  be the concatenation of  $s$  and  $i$ .

The set  $2^{<\omega}$  is a (binary) tree with respect to the partial order:  $s \leq t$  if  $s = t|n$  for some  $n \leq |t|$ . The empty sequence is the smallest element of  $2^{<\omega}$ .

Let  $x_\emptyset^* = 0$  and  $x_\emptyset = 0$ . By induction on the tree  $2^{<\omega}$ , we shall construct sequences  $(x_t^*)_{t \in 2^{<\omega}} \subset D$  and  $(x_t)_{t \in 2^{<\omega}} \subset X$  such that for every  $t \in 2^{<\omega}$  the following conditions are satisfied:

- (1)  $x_{t0}^* = x_t^*$  and  $x_{t0} = x_t$ ;
- (2)  $|x_{t1}^*(x_s) - x_t^*(x_s)| < 2^{-|t|}\varepsilon$  for all  $s \in 2^{<\omega}$  with  $|s| \leq |t|$ ;
- (3)  $\|x_{t1}\| = 1$ ;
- (4)  $x_{t1}^*(x_{t1}) - x_t^*(x_{t1}) \geq 4\varepsilon$ .

Suppose for some  $t \in 2^{<\omega}$  the functionals  $x_s^*$  and points  $x_s$  have been constructed for all  $s \in 2^{<\omega}$  with  $|s| < |t|$ . If  $t = \tau 0$  for some  $\tau \in 2^{<\omega}$ , then we put  $x_t^* = x_\tau^*$  and  $x_t = x_\tau$ .

Now consider the other case:  $t = \tau 1$  for some  $\tau \in 2^{<\omega}$ . Consider the weak\* open set

$$W = \{x^* \in D : \forall s \in 2^{<\omega} \ |s| < |t| \Rightarrow |x^*(x_s) - x_\tau^*(x_s)| < \varepsilon\}$$

in  $D$ . Since  $W \neq \emptyset$ , we have  $\text{diam } W > 8\varepsilon$ . Consequently there exists a functional  $x_t^* \in W$  such that  $\|x_t^* - x_\tau^*\| > 4\varepsilon$ . Choose a point  $x_t \in X$  with  $\|x_t\| = 1$  and  $(x_t^* - x_\tau^*)(x_t) \geq 4\varepsilon$ . This completes the inductive construction.

For every  $i \in \mathbb{N}$  let

$$U_i = \{x^* \in X^* : |x^*(x_s)| < \varepsilon \text{ for every } s \in 2^{<\omega} \text{ with } |s| \leq i\}.$$

Evidently,  $U_i$  are weak\* open sets in  $X^*$  and their intersection  $\bigcap_{i=1}^{\infty} U_i$  contains the open  $\varepsilon$ -ball  $\varepsilon B^*$ . To see that W\*CP fails for the space  $X^*$  it suffices to check that  $B^* \not\subset \bigcup_{i=1}^n a_i^* + U_i$  for every  $n \in \mathbb{N}$  and points  $a_1^*, \dots, a_n^* \in X^*$ . This will follow as soon as we find  $t \in 2^n$  with  $x_t^* \notin \bigcup_{j=1}^n (a_j^* + U_j)$ .

Since  $|(x_1^* - x_0^*)(x_1)| \geq 4\varepsilon$ , there is  $t_0 \in \{0, 1\}$  with  $|x_{t_0}^*(x_1) - a_1^*(x_1)| \geq 2\varepsilon$ . By the same reason, the inequality  $(x_{(t_0,1)} - x_{(t_0,0)})(x_{(t_0,1)}) \geq 4\varepsilon$  yields a number  $t_1 \in \{0, 1\}$  such that  $|(x_{(t_0,t_1)}^* - a_2^*)(x_{(t_0,1)})| \geq 2\varepsilon$ . Proceeding by finite induction and using (4), we may construct a sequence  $t = (t_0, t_1, \dots, t_{n-1}) \in 2^n$  such that for every  $k \leq n$

$$(5) \quad |(x_{t|k}^* - a_k^*)(x_{s_k})| \geq 2\varepsilon \text{ for some sequence } s_k \in 2^k.$$

Let us show that  $x_t^* \notin \bigcup_{j=1}^n (a_j^* + U_j)$ . Assuming the converse, we would find a number  $p \leq n$  with  $x_t^* - a_p^* \in U_p$  which implies

$$(6) \quad |(x_t^* - a_p^*)(x_{s_p})| < \varepsilon.$$

It follows from (2) that

$$|(x_t^* - x_{t|p}^*)(x_{s_p})| \leq \sum_{k=p}^{n-1} |(x_{t|k+1}^* - x_{t|k}^*)(x_{s_p})| \leq \sum_{k=p}^{n-1} 2^{-k} \varepsilon < 2^{-p+1} \varepsilon \leq \varepsilon$$

which together with (6) yields the inequality  $|(x_{t|p}^* - a_p^*)(x_{s_p})| < 2\varepsilon$  that contradicts (5).  $\square$

### 3. Proof of Theorem 1

The following two lemmas yield the “ $\infty$ -reflexive” part of Theorem 1.

**Lemma 4.** *Each net in an infinite-dimensional  $\infty$ -reflexive Banach space has an accumulation point in the weak topology.*

PROOF: Assume that some  $\varepsilon$ -net  $N$  in  $X$  has no accumulating points in the weak topology. Replacing  $N$  by a suitable homothetic copy, we can assume that  $\varepsilon = \frac{1}{8}$ . Since  $N$  has no accumulation points in the weak topology, there is a cover  $\mathcal{U}$  of  $X$  by weakly open subsets such that each set  $U \in \mathcal{U}$  has at most one common point with the net  $N$ . Since  $X$  is  $\infty$ -reflexive, there is a finite subfamily  $\mathcal{V} \subset \mathcal{U}$  whose union  $\bigcup \mathcal{V}$  contains some ball  $B$  of radius 1. Then  $B \cap N \subset \bigcup_{V \in \mathcal{V}} V \cap N$  is finite. One can easily check that  $B \cap N$  is a  $\frac{1}{4}$ -net for  $B$ , which implies that  $X$  is finite-dimensional according to the classical Riesz Lemma on an almost orthogonal element, see [HHZ, Lemma 15].  $\square$

**Lemma 5.** *A separable Banach space  $X$  is  $\infty$ -reflexive if each net in  $X$  has an accumulation point in the weak topology.*

PROOF: Assuming that  $X$  is not  $\infty$ -reflexive, find a cover  $\mathcal{U}$  of  $X$  by weakly open sets such that for every finite subfamily  $\mathcal{V} \subset \mathcal{U}$  the union  $\bigcup \mathcal{V}$  contains no ball of radius 1. Using the separability of  $X$ , we can assume that the cover  $\mathcal{U}$  is countable and hence can be enumerated as  $\mathcal{U} = \{U_n : n \in \omega\}$ . Let  $\{a_n : n \in \omega\}$  be a countable dense set in  $X$ . For every  $n \in \omega$  we can find a point  $x_n \in X \setminus \bigcup_{i < n} U_i$  with  $\|x_n - a_n\| \leq 1$ . Such a point  $x_n$  exists by the choice of the cover  $\mathcal{U}$ . Then  $\{x_n : n \in \omega\}$  is a 2-net in  $X$  having no accumulation point in the weak topology.  $\square$

The “ $\omega$ -reflexive” part of Theorem 1 is established in the following more general characterization of the  $\omega$ -reflexivity. However we shall need a more general meaning for an  $\varepsilon$ -net: a subset  $N$  of a Banach space  $(X, \|\cdot\|)$  is called an  $\varepsilon$ -net for a subset  $B \subset X$  if for every  $x \in B$  there is  $y \in N$  with  $\|x - y\| < \varepsilon$ .

**Lemma 6.** *For a separable infinite-dimensional Banach space  $X$  the following conditions are equivalent:*

- (1)  $X$  is  $\omega$ -reflexive;
- (2) each net for  $X$  contains a non-trivial sequence convergent in the weak topology of  $X$ ;
- (3) there are a bounded set  $D \subset X$  and  $\varepsilon > 0$  such that each  $\varepsilon$ -net  $N \subset X$  for  $D$  has an accumulation point in the weak topology of  $X$ .

PROOF: We shall prove the equivalences (1)  $\Leftrightarrow$  (3)  $\Leftrightarrow$  (2).

(1)  $\Rightarrow$  (3). Assume that  $X$  is  $\omega$ -reflexive and find  $r \in \mathbb{N}$  such that  $X$  is  $r$ -reflexive. We claim that each  $\frac{1}{4}$ -net for the ball  $(r+1)B = \{x \in X : \|x\| < r+1\}$  has an accumulating point in the weak topology of  $X$ . Assuming that it is not so, find a  $\frac{1}{4}$ -net  $N \subset X$  for  $(r+1)B$  having no accumulation point in the weak topology. This allows us to construct a cover  $\mathcal{U}$  of  $X$  by weakly open sets having at most one-point intersection with the net  $N$ . The  $r$ -reflexivity of  $X$  yields a finite subfamily  $\mathcal{V} \subset \mathcal{U}$  covering the 1-ball  $x+B = \{y \in X : \|x-y\| < 1\}$  centered at a point  $x \in X$  with  $\|x\| \leq r$ . Then the intersection  $(x+B) \cap N \subset \bigcup_{V \in \mathcal{V}} V \cap N$  is finite and thus lies in a finite-dimensional subspace  $F \subset X$ . The Riesz almost orthogonality Lemma 15 in [HHZ] allows us to find a point  $y \in B_x$  such that  $\|y - x\| = \frac{1}{2}$  but  $\text{dist}(y, F) > \frac{1}{4}$ . Using the fact that  $N$  is a  $\frac{1}{4}$ -net for  $(r+1)B \supset x+B$ , find a point  $z \in N$  with  $\|z - y\| < \frac{1}{4}$ . Then  $z \in (x+B) \cap N \setminus F$  which is not possible because  $(x+B) \cap N \subset F$ .

(3)  $\Rightarrow$  (1) Assume that for some bounded set  $D \subset X$  and some  $\varepsilon > 0$  each  $\varepsilon$ -net  $N \subset X$  for  $D$  has an accumulation point in the weak topology. Replacing  $D$  by its homothetic copy, we can assume that  $\varepsilon = 1$ . Let  $r = \sup\{\|x\| : x \in D\}$ . We claim that the space  $X$  is  $r$ -reflexive. Otherwise we can find an open cover  $\mathcal{U}$  of

$X$  by weakly open subsets such that no finite subfamily of  $\mathcal{U}$  covers the open unit ball centered at a point  $x \in X$  with  $\|x\| \leq r$ . Use the separability of  $X$  to find a countable subcover  $\{U_n : n \in \omega\} \subset \mathcal{U}$  of  $X$  and let  $\{x_n : n \in \omega\}$  be a countable dense set in the  $r$ -ball  $rB = \{x \in X : \|x\| < r\}$ . For every  $n \in \omega$  select a point  $y_n \in X \setminus \bigcup_{k < n} U_k$  with  $\|x_n - y_n\| < 1$  (such a point  $y_n$  exists by the choice of the cover  $\mathcal{U}$ ). Then  $N = \{y_n : n \in \omega\}$  is a 2-net for  $D$  without accumulation points in the weak topology of  $X$ . This is a contradiction.

(3)  $\Rightarrow$  (2) Assume that for some bounded set  $D \subset X$  and some  $\varepsilon > 0$  each  $\varepsilon$ -net  $N \subset X$  for  $D$  has an accumulation point in the weak topology. Then given any  $\varepsilon$ -net  $N$  in  $X$  we can find a bounded subset  $A \subset N$  having an accumulation point  $a \in X$  in the weak topology of  $X$ . The implication (3)  $\Rightarrow$  (1) ensures that  $X$  is  $\omega$ -reflexive and hence  $\infty$ -reflexive. By Corollary 1, the bounded subset  $A \cup \{a\}$ , being Fréchet-Urysohn, contains a sequence  $\{a_n\}_{n=1}^{\infty} \subset A \setminus \{a\}$  that converges to  $a$ .

(2)  $\Rightarrow$  (3). Assume conversely that for each bounded set  $D$  and every  $\varepsilon > 0$  there is an  $\varepsilon$ -net  $N \subset X$  for  $D$  having no accumulation point in the weak topology of  $X$ . In particular, for every  $r \in \omega$ , there is an 1-net  $N_r$  for the  $r$ -ball  $B_r = \{x \in X : \|x\| \leq r\}$  having no accumulating point in the weak topology of  $X$ . Now consider the union  $N = \bigcup_{r \in \omega} N_r \setminus B_{r-2}$  and note that it is an 1-net for  $X$ . Indeed, given any  $x \in X$  find  $r \in \omega$  with  $r - 1 < \|x\| \leq r$  and  $y \in N_r$  with  $\|x - y\| < 1$ . Then  $\|y\| > \|x\| - 1 > r - 2$  and hence  $y \in N_r \setminus B_{r-2} \subset N$ . Assuming that  $N$  contains a non-trivial weakly convergent sequence  $S \subset N$ , find  $R \in \omega$  with  $S \subset B_{R-2}$  and observe that  $S \subset N \cap B_{R-2} \subset \bigcup_{r \leq R} N_r$ . Then for some  $r \leq R$  the intersection  $S \cap N_r$  is infinite and hence  $N_r \supset S \cap N_r$  has an accumulation point in the weak topology, which contradicts the choice of  $N_r$ .  $\square$

#### 4. Proof of Proposition 1

Let  $Z$  be a subspace of a Banach space  $X$  and let  $\pi : X \rightarrow X/Z$  denote the quotient operator.

1. Assuming that  $X$  is  $r$ -reflexive for some  $r \in [0, +\infty]$ , we shall prove that the quotient space  $X/Z$  is  $r$ -reflexive too. Given a cover  $\mathcal{U}$  of  $X/Z$  by weakly open sets, consider the cover  $\pi^{-1}(\mathcal{U}) = \{\pi^{-1}(U) : U \in \mathcal{U}\}$  of  $X$ . By the  $r$ -reflexivity of  $X$  there is a finite subfamily  $\mathcal{V} \subset \mathcal{U}$  whose preimage  $\pi^{-1}(\mathcal{V})$  covers some ball  $x + B_X = \{y \in X : \|x - y\| < 1\}$  centered at a point  $x \in X$  with  $\|x\| \leq r$ . Then the family  $\mathcal{V}$  covers the image  $\pi(x + B_X)$  which coincides with the ball  $\pi(x) + B_{X/Z} = \{z \in X/Z : \|z - \pi(x)\| < 1\}$  according to the definition of the quotient norm on  $X/Z$ . Taking into account that  $\|\pi(x)\| \leq \|x\| \leq r$ , we conclude that the space  $X/Z$  is  $r$ -reflexive.

2. Assume that  $X$  is  $\omega$ -reflexive and a Banach space  $Y$  is isomorphic to  $X$ . Let  $T : X \rightarrow Y$  be an isomorphism between  $X$  and  $Y$  and  $M = \max\{\|T\|, \|T^{-1}\|\}$ .

Let  $B_X, B_Y$  denote the open unit balls centered at the origins of the spaces  $X, Y$ , respectively. It follows that  $\frac{1}{M}B_Y \subset T(B_X) \subset M \cdot B_Y$ .

The space  $X$ , being  $\omega$ -reflexive, is  $r$ -reflexive for some  $r$ . We claim that  $Y$  is  $M^2r$ -reflexive. Indeed, given a cover  $\mathcal{U}$  of  $Y$  by weakly open sets, consider the covers  $\mathcal{W} = T^{-1}(\mathcal{U}) = \{T^{-1}(U) : U \in \mathcal{U}\}$  and  $\frac{1}{M}\mathcal{W} = \{\frac{1}{M} \cdot W : W \in \mathcal{W}\}$  of  $X$ . The  $r$ -reflexivity of  $X$  implies the existence of a finite subfamily  $\mathcal{V} \subset \mathcal{U}$  such that  $\bigcup_{V \in \mathcal{V}} \frac{1}{M}T^{-1}(V)$  covers the unit ball  $x + B_X$  centered at some point  $x \in X$  with  $\|x\| \leq r$ . Letting  $y = M \cdot T(x)$ , observe that  $\|y\| = M \cdot T(x) \leq M^2r$  and  $y + B_Y \subset T(Mx + MB_X) = M \cdot T(x + B_X) \subset M \cdot T(\bigcup_{V \in \mathcal{V}} \frac{1}{M}T^{-1}(V)) = \bigcup \mathcal{V}$ , witnessing the  $M^2r$ -reflexivity of the space  $Y$ .

By analogy we can prove that the  $\infty$ -reflexivity of  $X$  implies the  $\infty$ -reflexivity of  $Y$ . Finally the 0-reflexivity coincides with the usual reflexivity and also is preserved by isomorphisms.

3. Assume that the space  $Z$  is reflexive and  $X/Z$  is  $r$ -reflexive for some  $r \in [0, \infty)$ . Since the short sequence  $0 \rightarrow Z \rightarrow X \rightarrow X/Z \rightarrow 0$  is exact, so is the sequence  $0 \rightarrow Z^{**} \rightarrow X^{**} \rightarrow (X/Z)^{**} \rightarrow 0$ , see [CG, 2.2.d]. Consequently, the second dual  $\pi^{**} : X^{**} \rightarrow (X/Z)^{**}$  of the quotient operator  $\pi : X \rightarrow X/Z$  has  $Z^{**} = Z$  as the kernel.

We claim that for each bounded weakly closed subset  $F \subset X$  the image  $\pi(F)$  is weakly closed in  $X/Z$ . It follows that the closure  $\bar{F}$  of  $F$  in the weak\*-topology of  $X^{**}$  is compact and so is its image  $\pi^{**}(\bar{F}) \subset (X/Z)^{**}$ . We claim that  $\pi(F) = \pi^{**}(\bar{F}) \cap \pi(X)$  which will ensure that  $\pi(F)$  is closed in  $X/Z$ . Indeed, the inclusion  $\pi(F) = \pi^{**}(\bar{F}) \cap \pi(X) \subset \pi^{**}(\bar{F}) \cap \pi(X)$  is trivial. To prove the reverse inclusion, take any point  $y^{**} \in \pi^{**}(\bar{F}) \cap \pi(X)$  and find two points  $x^{**} \in \bar{F}$  and  $x \in X$  with  $\pi^{**}(x^{**}) = \pi(x) = y^{**}$ . It follows that  $x^{**} - x \in \text{Ker}(\pi^{**}) = Z^{**} = Z$  and hence  $x^{**} \in x + Z \subset X$ . Now we see that  $x^{**} \in \bar{F} \cap X = F$  and hence  $y^{**} = \pi^{**}(x^{**}) \in \pi^{**}(F) = \pi(F)$ .

Since the quotient homomorphism  $\pi$  maps bounded weakly closed subsets of  $X$  to bounded weakly closed sets of  $X/Z$ , the image  $\pi(\bar{B}_X)$  of the closed unit ball centered at the origin of  $X$  coincides with the closed unit ball  $\bar{B}_{X/Z}$  centered at the origin of  $X/Z$ .

Now we are ready to show that the space  $X$  is  $r$ -reflexive. Take any weakly open cover  $\mathcal{U}$  of  $X$ . For every point  $y \in (r+1)\bar{B}_{X/Z}$  the set  $(r+1)\bar{B}_X \cap \pi^{-1}(y)$  is weakly compact and hence can be covered by a finite subfamily  $\mathcal{U}_y \subset \mathcal{U}$ . The set  $F_y = (r+1)\bar{B}_X \setminus \bigcup \mathcal{U}_y$  is bounded and weakly closed in  $X$ . Consequently, its projection  $\pi(F_y)$  is weakly closed in  $X/Z$  while the complement  $V_y = (r+1)\bar{B}_{X/Z} \setminus \pi(F_y)$  is a weakly open neighborhood of  $y$  in  $(r+1)\bar{B}_{X/Z}$ . Since the space  $X/Z$  is  $r$ -reflexive the cover  $\{V_y : y \in (r+1)\bar{B}_{X/Z}\}$  of the closed ball  $(r+1)\bar{B}_{X/Z}$  contains a finite subcollection  $\{V_{y_1}, \dots, V_{y_n}\}$  whose union contains the open 1-ball  $y + B_{X/Z}$  centered at some point  $y \in X/Z$  with  $\|y\| \leq r$ . Take any point  $x \in X$  with  $\|x\| = \|y\|$  and  $\pi(x) = y$  and observe that  $\mathcal{W} = \bigcup_{i=1}^n \mathcal{U}_{y_i}$  is a finite cover of the

open 1-ball  $x + B_X$  centered at  $x$ . This witnesses that the space  $X$  is  $r$ -reflexive.

## 5. Proof of Theorem 2

In this section we prove that the James space  $J$  fails to be  $\omega$ -reflexive. We recall that the James space  $J$  is the Banach space consisting of all real sequences  $(x_n)_{n \in \omega}$  that tend to zero and have norm

$$\|(x_i)\| = \sup_{n_0 < \dots < n_k} \sqrt{\sum_{i=1}^k (x_{n_i} - x_{n_{i-1}})^2} < \infty.$$

Let  $J_0$  denote the set of all eventually zero sequences.

First we prove

**Lemma 7.** *For every  $M > 0$  there is  $\varepsilon > 0$  such that for every  $x \in J_0$  with  $\|x\| \leq M$  there is  $y = (y_n) \in J$  such that  $\|x - y\| < 1$  and  $|y_n - 1| \geq \varepsilon$  for all  $n \in \omega$ .*

PROOF: Given  $M > 0$  find an integer  $m \geq 2$  with  $\frac{20M}{\sqrt{2m+1}} < \frac{1}{2}$  and  $4M^2(2m+1) > 1$ , and let  $\varepsilon = \frac{1}{4m+2}$ .

Take any point  $x = (x_n) \in J_0$  with  $\|x\| \leq M$ . By induction, construct an increasing finite number sequence  $(k_i)_{i=0}^r$  such that for  $k_{r+1} = \infty$  we get

- $k_0 = 0$ ;
- $|x_p - x_q| \leq \varepsilon$  for all numbers  $p, q \in [k_i, k_{i+1})$  and all  $0 \leq i \leq r$ ;
- for every  $0 < i \leq r$  there is a number  $p_i \in [k_{i-1}, k_i)$  with  $|x_{k_i} - x_{p_i}| > \varepsilon$ .

It follows that

$$M \geq \|x\| \geq \sqrt{\sum_{0 < i \leq r} |x_{k_i} - x_{p_i}|^2} > \sqrt{r\varepsilon^2}$$

and hence  $r < \frac{M^2}{\varepsilon^2}$ . Let  $A = 2\varepsilon \cdot \mathbb{Z}$  be the arithmetic progression with step  $2\varepsilon$  and let  $f : \mathbb{R} \rightarrow A$  be a function assigning to each real number  $t \in \mathbb{R}$  a number  $f(t) \in A$  with  $|t - f(t)| \leq \varepsilon$ . Given a number  $a \in A$ , let

$$r_a = |\{i \leq r : f(x_{k_i}) \in \{a - 2\varepsilon, a, a + 2\varepsilon\}\}|.$$

Since  $|A \cap [\frac{1}{2}, \frac{3}{2}]| = \frac{1}{2\varepsilon} = 2m+1$ , there is a point  $a \in A \cap [\frac{1}{2}, \frac{3}{2}]$  with  $r_a \leq \frac{3r}{2m+1} \leq \frac{3M^2}{(2m+1)\varepsilon^2} = 12M^2(2m+1)$ . Taking into account that  $1 < 4M^2(2m+1)$ , we get  $r_a + 1 \leq 16M^2(2m+1)$ .

Consider the sequence  $z = (z_n)_{n \in \omega}$  such that  $z_n = 0$  for  $n \in [k_r, \infty)$  and for every  $i < r$  and  $n \in [k_i, k_{i+1})$  we have

$$z_n = \begin{cases} 0 & \text{if } f(x_{k_i}) \notin \{a - 2\varepsilon, a, a + 2\varepsilon\}; \\ -5\varepsilon & \text{if } f(x_{k_i}) = a - 2\varepsilon; \\ 5\varepsilon & \text{if } f(x_{k_i}) \in \{a, a + 2\varepsilon\}. \end{cases}$$

The definition of the norm on the James space  $J$  implies that

$$\begin{aligned} \|z\| &\leq \sqrt{(r_a + 1)(10\varepsilon)^2} \leq \sqrt{16M^2(2m + 1)100\varepsilon^2} \\ &= \sqrt{\frac{1600M^2}{4(2m + 1)}} = \frac{20M}{\sqrt{2m + 1}} < \frac{1}{2}. \end{aligned}$$

Let  $e = (e_n)_{n \in \omega}$  be the element of  $J$  such that  $e_n = 1$  for all  $i < k_r$  and  $e_n = 0$  for all  $n \geq k_r$ . It is clear that  $\|e\| = 1$ .

Finally, consider the point  $y = x + z + (1 - a) \cdot e$ . Observe that

$$\|y - x\| = \|z + (1 - a) \cdot e\| \leq \|z\| + |1 - a| \cdot \|e\| < \frac{1}{2} + \frac{1}{2} = 1.$$

Now, we show that  $|y_n - 1| \geq \varepsilon$  for all  $n \in \omega$ . Indeed, if  $n \geq k_r$ , then  $y_n = x_n$  and  $|y_n - 1| \geq 1 - |x_n| \geq 1 - \varepsilon \geq \varepsilon$ .

Next, assume that  $n \in [k_i, k_{i+1})$  for some  $i < r$ . If  $f(x_{k_i}) \notin \{a - 2\varepsilon, a, a + 2\varepsilon\}$ , then

$$\begin{aligned} |y_n - 1| &= |x_n + z_n + (1 - a) - 1| = |x_n - a| = |x_n - f(x_{k_i}) + f(x_{k_i}) - a| \\ &\geq |f(x_{k_i}) - a| - |x_{k_i} - f(x_{k_i})| - |x_n - x_{k_i}| \geq 4\varepsilon - \varepsilon - \varepsilon \geq \varepsilon. \end{aligned}$$

If  $f(x_{k_i}) = a - 2\varepsilon$ , then

$$\begin{aligned} |y_n - 1| &= |x_n + z_n + (1 - a) - 1| = |x_n - x_{k_i} + x_{k_i} + f(x_{k_i}) - f(x_{k_i}) + z_n - a| \\ &\geq |z_n + f(x_{k_i}) - a| - |x_n - x_{k_i}| - |f(x_{k_i}) - x_{k_i}| \geq 3\varepsilon - \varepsilon - \varepsilon = \varepsilon. \end{aligned}$$

The case  $f(x_{k_i}) \in \{a, a + 2\varepsilon\}$  can be considered by analogy.  $\square$

The following lemma combined with Lemma 6 implies that the James space is not  $\omega$ -reflexive.

**Lemma 8.** *For every  $R \in \mathbb{N}$  the ball  $B_R = \{x \in J : \|x\| \leq R\}$  possesses a 2-net in  $J$  which is closed and discrete in the weak topology of  $J$ .*

PROOF: Using Lemma 7, find  $\varepsilon > 0$  such that the set

$$A_\varepsilon = \{(y_n) \in J : |y_n - 1| \geq \varepsilon \text{ for all } n \in \omega\}$$

intersects each open ball of unit radius centered at a point  $x \in J_0$  with  $\|x\| \leq R$ . Fix a countable dense set  $D = \{x_n : n \in \omega\}$  in  $A_\varepsilon \cap B_{R+1}$ . It follows that  $D$  is a 1-net for the ball  $B_R$ . For every  $n \in \omega$  consider the sequence  $\vec{e}_n = (1, \dots, 1, 0, \dots)$  with first  $n$  units. Since  $\|e_n\| = 1$  for all  $n \in \omega$ , the set  $D' = \{x_n - e_n : n \in \omega\}$  is a 2-net for the ball  $B_R$  in  $J$ . We claim that  $D'$  is closed and discrete in the weak topology of  $J$ . Assuming the converse and using the metrizability of the weak topology of  $J$  on bounded subsets, find an increasing number sequence  $(n_k)$  such that the sequence  $(x_{n_k} - e_{n_k})_{k \in \omega}$  weakly converges to some point  $z \in J$ . The weak convergence implies the coordinate convergence. Now it is convenient to think of the elements of  $J$  as functions defined on  $\omega$ . It follows that for every  $i \in \omega$ ,  $|z(i)| = \lim_{k \rightarrow \infty} |x_{n_k}(i) - e_{n_k}(i)| = \lim_{k \rightarrow \infty} |x_{n_k}(i) - 1| \geq \varepsilon$ , which is not possible because  $\lim_{i \rightarrow \infty} z(i) = 0$ .  $\square$

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