The Reissner-Nordström metric

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March 16, 2016

Abstract

A brief review of special and general relativity including some classical electrodynamics is given. We then present a detailed derivation of the Reissner-Nordström metric. The derivation is done by solving the Einstein-Maxwell equations for a spherically symmetric electrically charged body. The physics of this spacetime is then studied. This includes gravitational time dilation and redshift, equations of motion for both massive and massless non-charged particles derived from the geodesic equation and equations of motion for a massive charged particle derived with lagrangian formalism. Finally, a quick discussion of the properties of a Reissner-Nordström black hole is given.

Contents

1 Introduction

In 1915 Einstein completed his general theory of relativity. It did not take long before the first non-trivial exact solution for the Einstein field equations was found by Karl Schwarzschild in 1916 which corresponds to the gravitational field of a spherically symmetric object [1]. In the same year Hans Reissner generalized Schwarzschild's solution to include an electrically charged object [2]. Gunnar Nordström later (independently of Reissner) arrived at the same solution [3] now known as the Reissner-Nordström metric.

The Reissner-Nordström metric is a famous solution to the Einstein field equations. It describes the spacetime geometry around a spherically non-rotating charged body. The universe at large appear to be electrically neutral, so it is highly unlikely to find a macroscopic object that possess a considerable amount of net charge. The Reissner-Nordström solution is therefore not relevant to realistic situations in astrophysics. It does however contribute in understanding the fundamental nature of space and time. Also, the Reissner-Nordström metric is a more general solution than the Schwarzschild metric, so by simply putting the electrical charge to zero we obtain the Schwarzschild solution (which has plenty of practical applications).

In this paper we will first give a short introduction to general relativity. Our main goal is then to present a detailed derivation of the Reissner-Nordström metric (which is often overlooked in many textbooks) without assuming a static spacetime. We then proceed to study some of the physics in this spacetime such as:

- Gravitational time dilation and redshift
- Equations of motion for non-charged particles (massive or massless) derived with the geodesic equation
- Equations of motion for charged massive particles derived with the Lagrangian formalism
- Event horizons and black holes

2 Review of Special Relativity

In this chapter we give an overview of special relativity. We do this since general relativity generalizes special relativity, giving a description of gravity as a geometric effect of space and time. Special relativity is a theory regarding the relationship between time and space. It is based on two postulates:

(1) The laws of physics take the same form in all inertial reference frames.

(2) The speed of light is the same for all observers.

With these assumptions it becomes necessary to replace the Galilean transformations of classical mechanics with the Lorentz transformations.

The Lorentz transformations predicts that events that occur at the same time for one observer does not occur at the same time for an observer that is moving relative to the first one. This means that the absolute time and space that is used in the Galilean transformations must be abandoned and we instead describe time and space as part of the same continuum know as *spacetime*. Other predictions is that moving object will be shortened, moving clocks run slower and that addition of velocities is not as simple as with Galilean transformations.

An event is an occurrence that is characterised by a definite time and location relative to a reference frame. This means that an event can be thought of as a point in spacetime. Consider an event in an inertial reference frame S that is given by the coordinates (ct, x, y, z), where c is the speed of light. Suppose now that there is another inertial reference frame S' with the spatial coordinate axes orientated as in S but is moving with constant velocity v relative to S in the x-direction. Let also the origins coincide at time zero. The coordinates of the event in S' is defined to be (ct', x', y', z') . The Lorentz transformations, which can easily be derived if one assumes that the transformations are linear, specifies that these coordinates have the relation

$$
\begin{aligned}
t' &= \gamma \left(t - \frac{vx}{c^2}\right) \\
x' &= \gamma \left(x - vt\right) \\
y' &= y \\
z' &= z
\end{aligned}
$$

where

$$
\gamma \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.
$$

These transformations of course reduce to the Galilean transformations if $v \ll c$, since then $\gamma \approx 1$ and $t \gg vx/c^2$.

In Euclidian space the distance between two points is invariant. In spacetime, the "distance" s that is invariant is

$$
s^2 = c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2.
$$

If we look at an infinitesimal spacetime interval (which of course also is an invariant) we have

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2.$$

Note that it is also possible to choose the interval to be

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2.$$

The choice of sign is arbitrary and has no physical implication as long as one is consistent. Throughout this paper we will use the sign convention of the former infinitesimal interval, i.e. $(+, -, -, -)$.

If we let $(x^0, x^1, x^2, x^3) \equiv (ct, x, y, z)$ we can write the infinitesimal interval as

$$ds^2 = \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3} \eta_{\alpha\beta} dx^\alpha dx^\beta,$$

where $\eta_{\alpha\beta}$ is the *metric tensor*. When we use a Cartesian coordinate system, the metric tensor is given by

$$\eta_{\alpha\beta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

and is known as the *Minkowski metric*. If we also use Einstein's summation convention, which implies summation over an index that appears both as a subscript and a superscript, we can write the infinitesimal interval as

$$ds^2 = \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3} \eta_{\alpha\beta} dx^\alpha dx^\beta \equiv \eta_{\alpha\beta} dx^\alpha dx^\beta. \tag{2.1}$$

Consider now again the Lorentz transformations. They can be written as

$$x'^\alpha = \Lambda^\alpha{}_\mu x^\mu,$$

where $\Lambda^\alpha{}_\mu$ is constant with the condition

$$\eta_{\alpha\beta} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta_{\mu\nu}.$$

The coordinate differentials transforms as

$$dx'^\alpha = \Lambda^\alpha{}_\mu dx^\mu,$$

and with the chain rule we can write that the coordinates transforms according to

$$x'^\alpha = \Lambda^\alpha{}_\mu x^\mu = \frac{\partial x'^\alpha}{\partial x^\mu} x^\mu.$$

2.1 4-vectors

In special relativity a *4-vector* is an object with four components that transform in a specific way under Lorentz transformations. More specifically, the components of a contravariant 4-vector V^α transforms according to

$$V^\alpha = \Lambda^\alpha{}_\mu V^\mu$$

while the components of a covariant vector V_α transforms as

$$V_\alpha = \Lambda_\alpha{}^\mu V_\mu.$$

Here we use a superscript to denote a contravariant index while a subscript denotes a covariant index.

With the metric tensor one can lower and raise indices. That is, we can change a contravariant vector to a covariant and vice versa:

$$\begin{aligned} V^\alpha &= \eta^{\alpha\mu} V_\mu \\ V_\alpha &= \eta_{\alpha\mu} V^\mu. \end{aligned}$$

Here $\eta^{\alpha\mu}$ is the inverse of $\eta_{\alpha\mu}$ and together they satisfy

$$\eta^{\alpha\mu}\eta_{\mu\beta} = \delta^\alpha_\beta,$$

where δ^α_β is the usual Kronecker delta defined by

$$\delta^\alpha_\beta = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases}$$

For example, given the contravariant vector $V^\alpha = (V^0, V^1, V^2, V^3)$ the corresponding covariant vector is

$$V_\alpha = (V_0, V_1, V_2, V_3) = (V^0, -V^1, -V^2, -V^3).$$

Raising and lowering indices does not only apply to vectors, but to tensors of any rank.

With the metric tensor we can define an invariant scalar product for 4-vectors according to

$$V^2 = \eta_{\alpha\beta} V^\alpha V^\beta = \eta^{\alpha\beta} V_\alpha V_\beta = V^\alpha V_\alpha.$$

In Euclidean space the scalar product of vectors is always nonzero. But scalar products of 4-vectors in spacetime can be be either positive, zero or negative. When $V^\alpha V_\alpha > 0$ we call it a *timelike* vector. When $V^\alpha V_\alpha = 0$ it is called a *null* vector. And lastly, when $V^\alpha V_\alpha < 0$ it is called a *spacelike* vector.

Consider now a moving particle. The proper time τ (i.e. the time measured by a clock following the particle) is independent of the coordinate system used. The infinitesimal interval is given by

$$\begin{aligned} d\tau &= \frac{ds}{c} = \frac{1}{c}\sqrt{dt^2 - dx^2 - dy^2 - dz^2} \\ &= \frac{1}{c}\sqrt{1 - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2} = dt\sqrt{1 - \frac{v^2}{c^2}} = \frac{dt}{\gamma}. \end{aligned}$$

Let $x^\alpha = x^\alpha(\tau) = (ct, \mathbf{r})$ be the trajectory in spacetime of the particle, where $\mathbf{r}$ is the position 3-vector given by $\mathbf{r} = (x, y, z)$. The 4-velocity u^α of the particle is defined by

$$u^\alpha \equiv \frac{dx^\alpha}{d\tau} = \left(c\frac{dt}{d\tau}, \frac{dt}{d\tau}\frac{d\mathbf{r}}{dt}\right) = \gamma(c, \mathbf{v}),$$

where $\mathbf{v} \equiv d\mathbf{r}/dt$ is the 3-velocity. In Newtonian mechanics the momentum $m\mathbf{v}$ is conserved. In relativistic mechanics it turns out that it is the 3-momentum defined by $\mathbf{p} \equiv \gamma m\mathbf{v}$ that is conserved. The 4-momentum is then defined by

$$p^\alpha \equiv mu^\alpha = m\gamma(c, \mathbf{v}) = (mc\gamma\,, \mathbf{p}),$$

where m is the rest mass of the particle. Let $\mathbf{F} \equiv d\mathbf{p}/dt$ be the 3-force. The kinetic energy of the particle is then given by

$$E_k = \int \mathbf{F}\cdot d\mathbf{x} = \int \frac{d\mathbf{p}}{dt}\cdot d\mathbf{x} = \int \mathbf{v}\cdot d\mathbf{p} = \int \mathbf{v}\cdot d(m\gamma\mathbf{v}).$$

Using integration by parts we obtain

$$\begin{aligned} E_k &= m\gamma v^2 - \int m\gamma\mathbf{v}\cdot d\mathbf{v} = m\gamma v^2 - \frac{m}{2}\int \gamma d(v^2) \\ &= m\gamma v^2 + \frac{mc^2}{\gamma} - E_0 = \gamma mc^2 - E_0, \end{aligned}$$

where E_0 is a constant of integration. By putting $v = 0$ so that $E_k = 0$ we can determine E_0 to be $E_0 = mc^2$. E_0 is interpreted as the "rest energy" while γmc^2 is interpreted as the total energy of the particle. This is an interesting result as it indicates that mass itself has energy content. This is famously known as the mass-energy equivalence.

Note that the first component in the 4-momentum was γmc, so if E is the total energy of the particle we can write the 4-momentum as

$$p^\alpha = (E/c\,, \mathbf{p}).$$

Finally we will derive the so-called relativistic energy-momentum relation. Since the inner product of 4-vectors is invariant, we must have that

$$p^\alpha p_\alpha = p'^\alpha p'_\alpha.$$

We can take the primed coordinate system to be an instantaneous rest frame. In that case we have $p'^\alpha = (mc, 0)$ while $p^\alpha = (E/c, \mathbf{p})$. With the notation $p = |\mathbf{p}|$, the above can then simply be rewritten as

$$E^2 = p^2c^2 + m^2c^4,$$

which is the energy-momentum relation.

2.2 Electrodynamics in Special Relativity

Maxwell's equations describe the generation and interaction of electric and magnetic fields with each other and by charges and currents. With the electric field $\mathbf{E}$, magnetic field $\mathbf{B}$, charge density ρ and current density $\mathbf{J}$, Maxwell's equations are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{2.2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.3}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.4}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \tag{2.5}$$

where ϵ_0 is the vacuum permittivity and μ_0 is the vacuum permeability that satisfy

$$c^2 = \frac{1}{\mu_0 \epsilon_0}.$$

If a particle has an electrical charge q, the electric and magnetic field will exert a force (the Lorentz force) given by

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right).$$

The conservation of charge can be written as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0. \tag{2.6}$$

This can easily be seen by considering a volume V with charge q. The net current that flows into V is

$$I = \frac{dq}{dt} = -\iint\limits_{\partial V} \mathbf{J} \cdot d\mathbf{S} = -\iiint\limits_{V} (\nabla \cdot \mathbf{J}) dV,$$

where we used the divergence theorem in the last step. The derivative of q can on the other hand be written as

$$\frac{dq}{dt} = \frac{d}{dt} \iiint\limits_V \rho dV = \iiint\limits_V \frac{\partial \rho}{\partial t} dV.$$

Comparing the two above equations we obtain equation (2.6).

The electric and magnetic fields can be expressed in terms of an electric scalar potential Φ and a magnetic vector potential $\mathbf{A}$. The electric and magnetic fields in terms of these potentials are given by

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} \tag{2.7}$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{2.8}$$

Maxwell's equations can now be formulated as

$$\nabla^2\Phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\epsilon_0} \tag{2.9}$$

$$\left(\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right) = -\mu_0 \mathbf{J}. \tag{2.10}$$

Though these equations look a bit more complicated, we have reduced the number of components to solve for from 6 to 4. Also, the potentials are not physically meaningful quantities that can be measured; the electric and magnetic field are. In other words, the potentials are not uniquely determined by Maxwell's equations. We can therefore make the replacements

$$\begin{aligned} \mathbf{A} &\to \mathbf{A} + \nabla\psi \\ \Phi &\to \Phi - \partial\psi/\partial t, \end{aligned}$$

and it will not affect the electric and magnetic field. For example, the electric field will not change since

$$\begin{aligned} \mathbf{E}' = -\nabla\phi' - \frac{\partial \mathbf{A}'}{\partial t} &= -\left(\nabla\phi - \nabla\frac{\partial\psi}{\partial t}\right) - \left(\frac{\partial \mathbf{A}}{\partial t} + \frac{\partial}{\partial t}\nabla\psi\right) \\ &= -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} = \mathbf{E}. \end{aligned}$$

To be able to make such replacements is know as *gauge freedom*, and these transformations are known as *gauge transformations*. An often used gauge is the so-called *Lorenz gauge*, which allows one to choose $\mathbf{A}$ and Φ so that they satisfy the condition

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0.$$

Using the d'Alembertian operator defined by $\Box \equiv \nabla^2 - c^{-2}\partial^2/\partial t^2$ and the Lorenz gauge, equation (2.9) and (2.10) can now be written as

$$\Box\,\Phi = -\rho/\epsilon_0$$
$$\Box\,\mathbf{A} = -\mu_0\mathbf{J}.$$

It is possible to write Maxwell's equations on a simple form (known as the *covariant formulation of electromagnetism*) that will be useful later. We do this by defining the *4-potential* A^α, the *electromagnetic field tensor* $F^{\alpha\beta}$ and the *4-current* j^α by

$$A^\alpha \equiv \left(\frac{\Phi}{c}, \mathbf{A}\right) \tag{2.11}$$

$$F^{\alpha\beta} \equiv A^{\beta,\alpha} - A^{\alpha,\beta} \tag{2.12}$$

$$j^\alpha \equiv (c\rho, \mathbf{J}), \tag{2.13}$$

where we have used the contravariant *4-gradient* defined by

$$A^{\beta,\alpha} \equiv \partial^\alpha A^\beta \equiv \frac{\partial}{\partial x_\alpha} A^\beta,$$

while the covariant 4-gradient is $\partial_\alpha = g_{\alpha\mu}\partial^\mu$. With the 4-potential the Lorenz gauge condition can now simply be written as

$$\partial_\alpha A^\alpha = 0.$$

The electromagnetic field tensor is antisymmetric ($F^{\alpha\beta} = -F^{\beta\alpha}$), and can in matrix form be written as

$$F_{\alpha\beta} = \begin{bmatrix} 0 & E_1/c & E_2/c & E_3/c \\ -E_1/c & 0 & -B_3 & B_2 \\ -E_2/c & B_3 & 0 & -B_1 \\ -E_3/c & -B_2 & B_1 & 0 \end{bmatrix}.$$

It is straightforward to show that Maxwell's equations can be written as

$$F^{\alpha\beta}{}_{,\beta} = -\mu_0\, j^\alpha \tag{2.14}$$

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0. \tag{2.15}$$

The first one of these corresponds to equations (2.2) and (2.5) while the second one is equivalent to (2.3) and (2.4).

The equation for charge conservation can also be written in a compact way. We simply do this by taking the 4-divergence of the 4-current and set it equal to zero:

$$j^\alpha{}_{,\alpha} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right) \cdot (c\rho, \mathbf{J}) = \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$

This is exactly the same as equation (2.6).

Another object, which will be very central in general relativity, is the *electromagnetic stress-energy tensor* given by [6]

$$T^{\alpha\beta} = \frac{1}{\mu_0}\left(\frac{1}{4}\eta^{\alpha\beta}F^{\mu\nu}F_{\mu\nu} - F^{\alpha\mu}F^{\beta}{}_{\mu}\right), \tag{2.16}$$

which describes the flow of electromagnetic energy and momentum in spacetime. Properties for the stress-energy tensor is that it is symmetric ($T^{\alpha\beta} = T^{\beta\alpha}$) and traceless:

$$T \equiv T^{\alpha}{}_{\alpha} = \eta_{\alpha\beta}T^{\alpha\beta} = \frac{1}{\mu_0}\left(\frac{1}{4}\eta_{\alpha\beta}\eta^{\alpha\beta}F^{\mu\nu}F_{\mu\nu} - \eta_{\alpha\beta}F^{\alpha\mu}F^{\beta}{}_{\mu}\right) \tag{2.17}$$

$$= \frac{1}{\mu_0}\left(\frac{1}{4}\eta_{\alpha\beta}\eta^{\alpha\beta}F^{\mu\nu}F_{\mu\nu} - F^{\alpha\mu}F_{\alpha\mu}\right) = 0, \tag{2.18}$$

since $\eta_{\alpha\beta}\eta^{\alpha\beta} = \eta^{\alpha}{}_{\alpha} = 4$.

3 Tensor Fields and Manifolds

In this chapter we review the foundations of the mathematical formulation that is used in general relativity. In general relativity spacetime is being treated as a 4-dimensional pseudo-Riemannian differentiable manifold, so here we go through these concepts.

An n-dimensional manifold is a set of points that resembles an n-dimensional Euclidean space near each point. Furthermore, points can be labeled by a system of n real-valued coordinates $x^\alpha = (x^1, x^2, \dots, x^n)$ such that there is a one-to-one correspondence between the points and the labels. The manifold does not need to be completely "covered" by one specific coordinate system. We can describe the manifold by a collection of coordinate systems where each coordinate system covers a subset of the manifold. At points on the manifold that are covered by two different coordinate system there is a set of equations that relates the coordinates of one system to the coordinates of the other. Suppose there is another coordinate system with the coordinates x'^α. At points on the manifold where x'^α and x^α overlap, each of these primed coordinates will be related by some function of the the non-primed coordinate. We can write this as

$$x'^\alpha = x'^\alpha(x^1, x^2, \dots, x^n).$$

Similarly, we can write each x^α as a function of the primed coordinates:

$$x^\alpha = x^\alpha(x'^1, x'^2, \dots, x'^n).$$

Note: in this section we let greek indices have the range $1, 2, \dots, n$. In later sections where we apply this to general relativity we will let greek indices have the range $0, 1, \dots, n-1$, where the 0-component corresponds to the time-component.

A differentiable manifold is a specific type of manifold. It essentially just means that we are able to do differential calculus on the manifold. More specifically, the partial derivatives $\partial x'^{\alpha}/\partial x^{\mu}$ and $\partial x^{\alpha}/\partial x'^{\mu}$ exist. These partial derivatives now specify the transformation between the two coordinate systems:

$$x'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} x^{\mu}, \qquad x^{\alpha} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} x'^{\mu}.$$

From now on we will only work with differentiable manifolds and we can therefore use the two synonymously.

A contravariant vector transform as

$$V'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} V^{\mu}. \tag{3.1}$$

while a covariant vector transform as

$$V'_{\alpha} = \frac{\partial x^{\mu}}{\partial x'^{\alpha}} V_{\mu}. \tag{3.2}$$

From the transformation rules of contravariant and covariant vectors we can generalize this to a tensor of any rank. Consider the direct product $T^{\alpha}{}_{\beta} \equiv V^{\alpha} U_{\beta}$ of one contravariant vector and one covariant vector. This object will transform as

$$T'^{\alpha}{}_{\beta} = \left(\frac{\partial x'^{\alpha}}{\partial x^{\mu}} V^{\mu} \right) \left(\frac{\partial x^{\nu}}{\partial x'^{\beta}} U_{\nu} \right) = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} T^{\mu}{}_{\nu}. \tag{3.3}$$

It is now easy to see the generalization; a tensor $T^{\alpha_1 \alpha_2 \dots \alpha_m}_{\beta_1 \beta_2 \dots \beta_n}$ will transform as

$$T'^{\alpha_1 \alpha_2 \dots \alpha_n}_{\beta_1 \beta_2 \dots \beta_m} = \frac{\partial x'^{\alpha_1}}{\partial x^{\mu_1}} \frac{\partial x'^{\alpha_2}}{\partial x^{\mu_2}} \cdots \frac{\partial x'^{\alpha_m}}{\partial x^{\mu_m}} \frac{\partial x^{\nu_1}}{\partial x'^{\beta_1}} \frac{\partial x^{\nu_2}}{\partial x'^{\beta_2}} \cdots \frac{\partial x^{\nu_n}}{\partial x'^{\beta_n}} T^{\mu_1 \mu_2 \dots \mu_n}_{\nu_1 \nu_2 \dots \nu_m}.$$

In special relativity the line element is given by $ds^2 = \eta_{\alpha\beta} dx^{\alpha} dx^{\beta}$. We will now use a more general (but still symmetric) metric tensor $g_{\alpha\beta}$ that will give the line element for the manifold:

$$ds^2 = g_{\alpha\beta} dx^{\alpha} dx^{\beta}.$$

We define the inverse metric tensor $g^{\alpha\beta}$ such that

$$g^{\alpha\beta} g_{\beta\gamma} = g^{\alpha}{}_{\gamma} = \delta^{\alpha}_{\gamma}.$$

It follows directly from the symmetry of $g_{\alpha\beta}$ that $g^{\alpha\beta}$ also is symmetric.

The *signature* is defined as the number of positive, negative and zero eigenvalues of the metric tensor $g_{\alpha\beta}$. For example, the Minkowski metric will have the signature $(+, -, -, -)$. A manifold with a metric with only positive signs is called *Riemannian*. A generalization of the Riemannian manifold is the *pseudo-Riemannian* manifold that do not need to have a signature with only positive signs. The signature used in describing spacetime is clearly not of only positive signs, and is therefore a pseudo-Riemannian manifold.

The metric tensor can be used to raise and lower indices:

$$g_{\alpha\beta}T^{\mu\beta}{}_{\nu} = T^{\mu}{}_{\alpha\nu} \,. \tag{3.4}$$

A tensor with only up-indices is called contravariant while a tensor with only down-indices is called covariant. If it has at least one up and one down index it is called mixed. Furthermore, a tensor with m up indices and n down indices will be denoted as an (m, n)-tensor.

A summation over an upper and lower index is called a contraction and will yield a new tensor of lower rank. For example, consider a tensor $T^{\alpha}{}_{\beta}{}^{\gamma}$. If we do a contraction of α with β we get a new tensor

$$R^{\gamma} \equiv T^{\alpha}{}_{\alpha}{}^{\gamma} = g_{\alpha\beta}T^{\alpha\beta\gamma}.$$

It is straightforward to show that R^{γ} satisfy the transformation law

$$R'^{\gamma} = \frac{\partial x'^{\gamma}}{\partial x^{\mu}} R^{\mu}.$$

The sum or difference of two tensors is a tensor of the same type. This can easily be seen if we look at a specific example of a sum between two second rank tensors. Consider the object $R^{\alpha}{}_{\beta} \equiv S^{\alpha}{}_{\beta} + T^{\alpha}{}_{\beta}$, where S and T are tensors. It will transform as

$$R^{\alpha}{}_{\beta} = S^{\alpha}{}_{\beta} + T^{\alpha}{}_{\beta} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} \left(S'^{\mu}{}_{\nu} + T'^{\mu}{}_{\nu} \right) \equiv \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} R'^{\mu}{}_{\nu}.$$

It is then obvious that this generalizes to a sum or difference between two tensors of any rank. It is now easy to convince oneself that any equation between two tensors with the same upper and lower indices will be invariant under coordinate transformations.

3.1 Covariant Differentiation and Christoffel Symbols

In general, the differentiation of a tensor does not yield another tensor. This can easily be seen by considering the contravariant vector V^{α} which transforms as

$$V'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} V^{\mu}.$$

Differentiating w.r.t. x'^{β} we get

$$\frac{\partial V'^{\alpha}}{\partial x'^{\beta}} = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} \frac{\partial V^{\mu}}{\partial x^{\nu}} + \frac{\partial^2 x'^{\alpha}}{\partial x^{\mu} \partial \nu} \frac{\partial x^{\nu}}{\partial x'^{\beta}} V^{\mu}.$$

Because of the second term on the right hand side, $\partial V^\alpha/\partial x^\beta$ does not transform as a tensor. A derivative that do yield another tensor when operating on a tensor is the *covariant derivative*, denoted by ∇_μ (or by use of a semicolon). The covariant derivative of a contravariant vector V^α is defined by

$$\nabla_\beta V^\alpha \equiv V^\alpha_{\ ;\beta} \equiv \partial_\beta V^\alpha + \Gamma^\alpha_{\beta\mu} V^\mu, \tag{3.5}$$

where $\Gamma^\alpha_{\beta\mu}$ is the *Christoffel symbols* (also known as affine connections) which can be thought of as a "correction" term. We claimed that $\nabla_\beta V^\alpha$ is a tensor, so it must transform as

$$\nabla'_\beta V'^\alpha = \frac{\partial x^\mu}{\partial x'^\beta}\frac{\partial x'^\alpha}{\partial x^\nu}\nabla_\mu V^\nu.$$

From this we can work out how the Christoffel symbols must transform. The both sides in the above equation can be expanded with the use of equation (3.5), then we transform V' to V:

$$\begin{aligned} LHS: \nabla'_\beta V'^\alpha &= \partial'_\beta V'^\alpha + \Gamma'^\alpha_{\beta\lambda} V'^\lambda \\ &= \frac{\partial x^\mu}{\partial x'^\beta}\frac{\partial x'^\alpha}{\partial x^\nu}\partial_\mu V^\nu + \frac{\partial x^\mu}{\partial x'^\beta}\frac{\partial^2 x'^\alpha}{\partial x^\mu \partial x^\nu} V^\nu + \Gamma'^\alpha_{\beta\lambda}\frac{\partial x'^\lambda}{\partial x^\rho} V^\rho \end{aligned}$$

$$RHS: \frac{\partial x^\mu}{\partial x'^\beta}\frac{\partial x'^\alpha}{\partial x^\nu}\nabla_\mu V^\nu = \frac{\partial x^\mu}{\partial x'^\beta}\frac{\partial x'^\alpha}{\partial x^\nu}\partial_\mu V^\nu + \frac{\partial x^\mu}{\partial x'^\beta}\frac{\partial x'^\alpha}{\partial x^\nu}\Gamma^\nu_{\mu\rho} V^\rho.$$

By comparing the LHS to the RHS one can reach the conclusion that the Christoffel symbols must transform as

$$\Gamma'^\alpha_{\beta\lambda} = \frac{\partial x^\mu}{\partial x'^\beta}\frac{\partial x^\rho}{\partial x'^\lambda}\frac{\partial x'^\alpha}{\partial x^\nu}\Gamma^\nu_{\mu\rho} - \frac{\partial x^\mu}{\partial x'^\beta}\frac{\partial x^\rho}{\partial x'^\lambda}\frac{\partial^2 x'^\alpha}{\partial x^\mu \partial x^\rho}.$$

From this transformation law we see that the affine connection is not a tensor. With similar reasoning one can show that the covariant derivative of a covariant vector is [8]

$$\nabla_\beta V_\alpha \equiv V_{\alpha;\beta} = \partial_\beta V_\alpha - \Gamma^\mu_{\alpha\beta} V_\mu\,,$$

and that for second order tensors it is

$$(2,0)\text{-tensor: } \nabla_\gamma T^{\alpha\beta} = \partial_\gamma T^{\alpha\beta} + \Gamma^\alpha_{\mu\gamma} T^{\mu\beta} + \Gamma^\beta_{\mu\gamma} T^{\alpha\mu} \tag{3.6}$$

$$(1,1)\text{-tensor: } \nabla_\gamma T^\alpha_{\ \beta} = \partial_\gamma T^\alpha_{\ \beta} + \Gamma^\alpha_{\mu\gamma} T^\mu_{\ \beta} - \Gamma^\mu_{\beta\gamma} T^\alpha_{\ \mu} \tag{3.7}$$

$$(0,2)\text{-tensor: } \nabla_\gamma T_{\alpha\beta} = \partial_\gamma T_{\alpha\beta} - \Gamma^\mu_{\alpha\gamma} T_{\mu\beta} - \Gamma^\mu_{\beta\gamma} T_{\alpha\mu}. \tag{3.8}$$

The covariant derivative is clearly a generalization of the partial derivative. One important distinction however is that the order of which covariant differentiations are done does matter (i.e. it does not commute). So in general, changing the order of covariant differentiation changes the result. Properties such as linearity

$$\nabla(T+S) = \nabla T + \nabla S$$

and product rule

$$\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$$

still holds.

If one require that the affine connection is *torsion free*, i.e. that it is symmetrical in its lower indices ($\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta}$), and that the covariant derivative of the metric tensor is zero everywhere (a property called *metric compatibility*), then the affine connection is unique [8]. Properties that directly follows from this is that the covariant derivative of the inverse metric is zero ($\nabla_\gamma g^{\alpha\beta} = 0$) and that it commutes with rasing and lowering indices:

$$g_{\alpha\beta}(\nabla_\gamma V^\beta) = \nabla_\gamma(g_{\alpha\beta}V^\beta) = \nabla_\gamma V_\alpha.$$

We can now derive an expression for the Christoffel symbols in terms of the metric tensor and its first derivatives. Consider the explicit expression of the covariant derivative of the metric tensor computed with equation (3.8):

$$0 = \nabla_\gamma g_{\alpha\beta} = \partial_\gamma g_{\alpha\beta} - \Gamma^{\lambda}_{\gamma\alpha}\, g_{\lambda\beta} - \Gamma^{\lambda}_{\gamma\beta}\, g_{\lambda\alpha}.$$

By doing three different permutations of the free indices and combining these equations one end up with

$$0 = \partial_\gamma g_{\alpha\beta} - \partial_\alpha g_{\beta\gamma} - \partial_\beta g_{\gamma\alpha} + 2\Gamma^{\lambda}_{\alpha\beta}\, g_{\lambda\gamma}.$$

Solving for the Christoffel symbol finally yields

$$\Gamma^{\lambda}_{\alpha\beta} = \frac{1}{2} g^{\lambda\mu}(\partial_\alpha g_{\beta\mu} + \partial_\beta g_{\mu\alpha} - \partial_\mu g_{\alpha\beta}). \tag{3.9}$$

3.2 Riemann Tensor

Consider the covariant derivative of a covariant vector:

$$\nabla_\beta V_\alpha = \partial_\beta V_\alpha - \Gamma^{\mu}_{\alpha\beta} V_\mu.$$

Since $\nabla_\beta V_\alpha$ is a $(0,2)$-tensor, according to equation (3.8), a second differentiation yields

$$\nabla_\gamma(\nabla_\beta V_\alpha) = \partial_\gamma(\nabla_\beta V_\alpha) - \Gamma^{\mu}_{\alpha\gamma}(\nabla_\beta V_\mu) - \Gamma^{\mu}_{\beta\gamma}(\nabla_\mu V_\alpha)$$

The three terms in the above expression can be written as

$$\partial_\gamma(\nabla_\beta V_\alpha) = \partial_\gamma \partial_\beta V_\alpha - \left(\partial_\gamma \Gamma^\mu_{\alpha\beta}\right) V_\mu - \Gamma^\mu_{\alpha\beta}\left(\partial_\gamma V_\mu\right)$$
$$\Gamma^\mu_{\alpha\gamma}\left(\nabla_\beta V_\mu\right) = \Gamma^\mu_{\alpha\gamma}\left(\partial_\beta V_\mu - \Gamma^\nu_{\mu\beta} V_\nu\right)$$
$$\Gamma^\mu_{\beta\gamma}\left(\nabla_\mu V_\alpha\right) = \Gamma^\mu_{\alpha\gamma}\left(\partial_\beta V_\mu - \Gamma^\nu_{\mu\beta} V_\nu\right).$$

Taking the differentiations in different order (i.e. interchanging β and γ) one can show that

$$\nabla_\gamma\left(\nabla_\beta V_\alpha\right) - \nabla_\beta\left(\nabla_\gamma V_\alpha\right) = V_\mu\left(\partial_\beta \Gamma^\mu_{\alpha\gamma} - \partial_\gamma \Gamma^\mu_{\alpha\beta} + \Gamma^\nu_{\alpha\gamma}\Gamma^\mu_{\nu\beta} - \Gamma^\nu_{\alpha\beta}\Gamma^\mu_{\nu\gamma}\right). \tag{3.10}$$

This leads us to make the definition

$$R^\mu{}_{\alpha\beta\gamma} \equiv \partial_\beta \Gamma^\mu_{\alpha\gamma} - \partial_\gamma \Gamma^\mu_{\alpha\beta} + \Gamma^\nu_{\alpha\gamma}\Gamma^\mu_{\nu\beta} - \Gamma^\nu_{\alpha\beta}\Gamma^\mu_{\nu\gamma}. \tag{3.11}$$

$R^\mu{}_{\alpha\beta\gamma}$ is called the *Riemann curvature tensor* (or simply *Riemann tensor*). That it really is a tensor can be understood by noting that the left hand side of equation (3.10) is a difference between two tensors and therefore is a tensor itself. Then the right hand side, $V_\mu R^\mu{}_{\alpha\beta\gamma}$, must of course also be a tensor. But since V_μ and $R^\mu{}_{\alpha\beta\gamma}$ are completely independent of each other we conclude that $R^\mu{}_{\alpha\beta\gamma}$ is a tensor.

Since the Christoffel symbols are constructed from the metric tensor and its first derivatives, the Riemann tensor is constructed from the metric tensor and its first and second derivatives. In fact, it turns out that the Riemann tensor is the *only* tensor that can be constructed from the metric tensor and its first and second derivatives [5].

If the Riemann tensor vanishes at each point on a manifold we can now say that the order of differentiations for a $(0, 1)$-tensor field does not matter. For other types of tensor fields we can make similar calculations as when deriving equation (3.10). This lets us state that the order of covariant differentiations of a tensor field of any rank and type does not matter if $R^\mu{}_{\alpha\beta\gamma} = 0$ at all points on the manifold. We can now also give a precise definition of curvature. If $R^\mu{}_{\alpha\beta\gamma} = 0$ at each point on a manifold, the manifold is flat. Otherwise it is curved.

One property of the Riemann tensor, known as the *cyclic identity*, is

$$R^\mu{}_{\alpha\beta\gamma} + R^\mu{}_{\beta\gamma\alpha} + R^\mu{}_{\gamma\alpha\beta} = 0,$$

which can be showed in a straightforward but tedious way. If we lower the upper index of the Riemann tensor, i.e. $R_{\alpha\beta\gamma\delta} \equiv g_{\alpha\mu} R^\mu{}_{\beta\gamma\delta}$, one can read off the following symmetrical properties [7]:

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$$
$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}$$
$$R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma}$$
$$R^\alpha{}_{\alpha\gamma\delta} = 0.$$

In a similar way as we got equation (3.10), one can show that the following holds for a second rank covariant tensor:

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) T_{\gamma\delta} = R^\mu{}_{\gamma\beta\alpha} T_{\mu\delta} + R^\mu{}_{\delta\beta\alpha} T_{\gamma\mu}. \tag{3.12}$$

From the definition of the Riemann tensor we had that

$$(\nabla_\gamma \nabla_\beta - \nabla_\beta \nabla_\gamma) V_\alpha = R^\mu{}_{\alpha\beta\gamma} V_\mu.$$

With the use of the product rule and the above equation we obtain

$$\nabla_\lambda (\nabla_\gamma \nabla_\beta - \nabla_\beta \nabla_\gamma) V_\alpha = V_\mu \nabla_\lambda R^\mu{}_{\alpha\beta\gamma} + R^\mu{}_{\alpha\beta\gamma} \nabla_\lambda V_\mu.$$

Using the above equation together with (3.12) (but with $\nabla_\gamma V_\beta$ instead of $T_{\gamma\beta}$) one can show that the Riemann tensor must satisfy the relation [9]

$$\nabla_\delta R^\mu{}_{\alpha\beta\gamma} + \nabla_\beta R^\mu{}_{\alpha\gamma\delta} + \nabla_\gamma R^\mu{}_{\alpha\delta\beta} = 0, \tag{3.13}$$

which is known as the *Bianchi identity*. In a n-dimensional space, the Riemann tensor has n^4 components. But with all the properties that the Riemann tensor possess, one can show that there only exist $n^2 \left(n^2 - 1\right)/12$ independent components [8]. For example, in a 4-dimensional manifold we have 20 independent components of the Riemann tensor.

From the Riemann tensor we can construct the *Ricci tensor*, which we define by

$$R_{\alpha\beta} \equiv R^\mu{}_{\alpha\mu\beta} = \partial_\mu \Gamma^\mu_{\alpha\beta} - \partial_\beta \Gamma^\mu_{\alpha\mu} + \Gamma^\mu_{\nu\mu} \Gamma^\nu_{\alpha\beta} - \Gamma^\mu_{\nu\beta} \Gamma^\nu_{\alpha\mu}. \tag{3.14}$$

From the cyclic identity of the Riemann tensor, it can be showed that the Ricci tensor is symmetric. The trace of the Ricci tensor is known as the *Ricci scalar*:

$$R \equiv R^\alpha{}_\alpha = g^{\alpha\beta} R_{\alpha\beta}. \tag{3.15}$$

Consider now the Bianchi identity given by equation (3.13). If we contract μ with β and use that $R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma}$ we obtain

$$\nabla_\delta R_{\alpha\gamma} + \nabla_\beta R^\beta{}_{\alpha\gamma\delta} - \nabla_\gamma R_{\alpha\delta} = 0.$$

Multiplying with $g^{\alpha\gamma}$ and using that

$$\begin{aligned} g^{\alpha\gamma} \nabla_\beta R^\beta{}_{\alpha\gamma\delta} &= \nabla_\beta R^{\beta\gamma}{}_{\gamma\delta} = -\nabla_\beta R^{\gamma\beta}{}_{\gamma\delta} \\ &= -\nabla_\beta \left(g^{\beta\alpha} R^\gamma{}_{\alpha\gamma\delta} \right) = -\nabla_\beta R^\beta{}_\delta \end{aligned}$$

we get

$$\nabla_\delta R - 2\, \nabla_\beta R^\beta{}_\delta = 0.$$

The above equation can be written in an equivalent form as

$$\nabla_\mu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = 0.$$

This expression leads us to define a new tensor, which is of great importance in general relativity, called the *Einstein tensor*:

$$G^{\alpha\beta} \equiv R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R.$$

The Einstein tensor is symmetric (since both $R^{\alpha\beta}$ and $g^{\alpha\beta}$ are symmetric) and divergence-less.

3.3 Parallel Transport and Geodesics

Parallel transport can be thought of as transporting a tensor along a curve on a manifold while keeping the tensor constant. For simplicity, consider a vector. On a flat manifold one can parallel transport the vector along a curve from one point to another and the result of this transport does not depend on how the curve looks like. This is very easy to see if one uses Cartesian coordinates. Then we can parallel transport a vector by simply keeping the components constant.

In a curved manifold it is not that simple anymore. The easiest way to understand this is probably to consider parallel transport of a vector on a 2-sphere. Imagine a vector at one point on the sphere. If one move this vector around a closed curve while letting it point in the "same direction" it will not be parallel to the "original" vector when it comes back to the starting point.

What this really means is that there is no well-defined way to globally say that two vectors are parallel or not. We can only compare two vectors if they are at the same point (i.e. they are elements of the same tangent space).

Now we will consider how to mathematically describe pararell transport. Consider first a flat manifold where a curve is given by $x^\mu(\lambda)$, where λ is a parameter. On this flat manifold the requirement that a tensor $T^{\alpha_1 \ldots \alpha_m}{}_{\beta_1 \ldots \beta_n}$ is constant along this curve is simply (with the chain rule)

$$0 = \frac{d}{d\lambda} T^{\alpha_1 \ldots \alpha_m}{}_{\beta_1 \ldots \beta_n} = \frac{dx^\mu}{d\lambda} \partial_\mu T^{\alpha_1 \ldots \alpha_m}{}_{\beta_1 \ldots \beta_n},$$

for all λ. This will obviously not hold for a curved manifold. The generalization of this is to simply change the partial derivative by a covariant derivative. So a tensor $T^{\alpha_1 \ldots \alpha_m}{}_{\beta_1 \ldots \beta_n}$ is said to be parallel transported along the curve $x^\mu(\lambda)$ if it for all λ satisfy

$$0 = \frac{dx^\mu}{d\lambda} \nabla_\mu T^{\alpha_1 \ldots \alpha_m}{}_{\beta_1 \ldots \beta_n}.$$

We will only consider the case when the tensor is a vector. For a contravariant vector V^α, the above equation reads

$$0 = \frac{dx^\mu}{d\lambda}\nabla_\mu V^\alpha = \frac{dx^\mu}{d\lambda}\left(\partial_\mu V^\alpha + \Gamma^\alpha_{\mu\nu} V^\nu\right) = \frac{dV^\alpha}{d\lambda} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\lambda}V^\nu. \tag{3.16}$$

We have now a clear definition of parallel transport. Next step is to talk about geodesics, which is a central part of general relativity. A geodesic can be thought of as the generalization of a straight line to curved spaces. We will define a geodesic to be a curve that parallel transport its own tangent vector. If one uses the Christoffel connection (as we do), another equivalent definition is to say that a geodesic is the path of the shortest distance between two points (which clearly is a generalization of a straight line) [8].

As before, let a curve be given by $x^\alpha(\lambda)$. The tangent vector along this curve is simply $dx^\alpha/d\lambda$. The condition that this tangent vector is parallel transported along the curve is given by (3.16). So if x^α is to be a geodesic it must satisfy

$$0 = \frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}.$$

This is called the *geodesic equation*. It is a nonlinear second order differential equation. If one has has an initial position $x^\alpha(\lambda_0)$ and direction $\frac{dx^\alpha}{d\lambda}\big|_{\lambda_0}$ it will give a unique geodesic.

4 Basics of General Relativity

With the basics of the mathematical foundation of general relativity described in the previous chapter, we now use it to discuss the physics of the theory. We start with the equivalence principle which is an important principle for generalizing the physics of special relativity to include gravity, and then introduce the principle of general covariance (which let us easily generalize electrodynamics to curved spacetime). Finally we show how motion in general relativity reduces to Newtonian mechanics and introduce the Einstein field equations.

4.1 The Equivalence Principle

The (strong) *equivalence principle* can be stated as:

> At every spacetime point in a gravitational field it is possible to choose a *locally inertial coordinate system* such that, within a sufficiently small region of the point, the laws of nature are the same as in special relativity (i.e. non-accelerated coordinate system in absence of gravitation).

There is great similarity between the equivalence principle and that a curved Riemannian manifold appears locally flat. Because of this resemblance one may expect that spacetime in general relativity can be described with a pseudo-Riemannian manifold.

In a locally inertial Cartesian coordinate system with coordinates L^α the metric is given by

$$ds^2 = \eta_{\alpha\beta} dL^\alpha dL^\beta.$$

By using that

$$dL^\alpha = \frac{\partial L^\alpha}{\partial x^\mu} dx^\mu$$

for any change to the arbitrary coordinates x^α, we have that

$$ds^2 = \tilde{g}_{\alpha\beta} dx^\alpha dx^\beta$$

where we defined

$$\tilde{g}_{\alpha\beta} \equiv \eta_{\mu\nu} \frac{\partial L^\mu}{\partial x^\alpha} \frac{\partial L^\nu}{\partial x^\beta}.$$

Locally in the coordinate system of L^α the equations of motion of a free particle is

$$\frac{d^2 L^\alpha}{d\lambda^2} = 0,$$

where λ is a parameter (for massive particles this parameter can be taken to be the proper time, but not for massless particles such as photons). Changing to the coordinates x^μ and using the chain rule we can write the equations of motion as

$$\frac{d^2 x^\alpha}{d\lambda^2} + \tilde{\Gamma}^\alpha_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0,$$

where we defined

$$\tilde{\Gamma}^\alpha_{\mu\nu} = \frac{\partial x^\alpha}{\partial L^\sigma} \frac{\partial^2 L^\sigma}{\partial x^\mu \partial x^\nu}.$$

This has exactly the same form as the geodesic equation derived in section 3.3. In fact, one can show that $\tilde{g}_{\alpha\beta}$ and $\tilde{\Gamma}^\alpha_{\mu\nu}$ has the exact same relation to each other as $g_{\alpha\beta}$ and $\Gamma^\alpha_{\mu\nu}$ has (as derived in section 3.1) [5]. So in general relativity, where we treat spacetime as a 4-dimensional pseudo-Riemannian manifold, we can express the equations of motion geometrically. That is, a particle in free fall will follow a path that is a geodesic, given by the geodesic equation.

4.2 The Principle of General Covariance

We will now discuss a very useful way of generalizing results that are valid in special relativity to be valid in general relativity known as the *principle of general covariance*. This principle states that a physical equation holds true in all coordinate systems if:

(1) The equation holds true in absence of gravitation (i.e it holds true in special relativity).

(2) It is a tensor equation (i.e it preserves its form under a general coordinate transformation).

By the equivalence principle one can write down an equation that holds in a locally inertial coordinate system and then make a general coordinate transformation to find the corresponding equation in that coordinate system. With the principle of general covariance finding the equations that holds for all coordinate systems is much simpler. It can be seen that it follows from the equivalence principle by considering any equation that satisfy condition (1) and (2). Since the equation is generally covariant it preserves its form under a general coordinate transformation, so if it is true in any coordinate system it is true in all coordinate systems. The equivalence principle tells us that at every point in spacetime there exists locally inertial coordinate systems in which the effects of gravity are absent. Since we assumed that our equation holds in special relativity (i.e. no gravity) and therefore holds in these locally inertial systems, it must hold in all coordinate systems.

Note that any equation can be made generally covariant by working out what it looks like in arbitrary coordinate systems. So in it self the principle of general covariance has no physical meaning.

Our method now to find equations that are valid in a general gravitational field is by simply take the valid equations (and definitions) of special relativity and replace partial derivatives (with respect to coordinates) by covariant derivative and the Minkowski metric $\eta_{\alpha\beta}$ by the general metric tensor $g_{\alpha\beta}$.

4.3 Electrodynamics in General Relativity

Recall that in special relativity, when no gravitational field is presence, Maxwell's equations can be written as

$$\begin{aligned}
&F^{\alpha\beta}{}_{,\beta} = -\mu_0\, j^{\alpha} \\
&F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0
\end{aligned}$$

if one is using a Cartesian coordinate system. When in a gravitational field these equations hold only for a locally inertial coordinate system at a point. But according to the principle of general covariance, if we simply change the partial derivatives to covariant derivatives the equations would hold for any coordinate system. That is, in a general coordinate system

Maxwell's equations are

$$F^{\alpha\beta}{}_{;\beta} = -\mu_0 \, j^\alpha$$
$$F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta} = 0.$$

The electromagnetic tensor (when in a coordinate basis) can still be defined by

$$F_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta}$$

since the Christoffel symbols is symmetric in the lower indices. Instead of raising and lowering indices with $\eta_{\alpha\beta}$, the more general metric tensor $g_{\alpha\beta}$ is now used. In special relativity the conservation of charge is expressed by $j^\alpha{}_{,\alpha} = 0$. In the same way as before, we change the derivative and this generalizes to

$$j^\alpha{}_{;\alpha} = 0.$$

Note that this only corresponds to a local conservation law of electric charge and not a global one. For the stress-energy tensor given by equation (2.16) in special relativity we only need to swap from the Minkowski metric tensor to the general metric tensor. The result is

$$T^{\alpha\beta} = \frac{1}{\mu_0}\left(\frac{1}{4} g^{\alpha\beta} F^{\mu\nu} F_{\mu\nu} - F^{\alpha\mu} F^\beta{}_\mu\right), \tag{4.1}$$

while it still is symmetric $\left(T^{\alpha\beta} = T^{\beta\alpha}\right)$ and traceless:

$$T \equiv T^\alpha{}_\alpha = g_{\alpha\beta} T^{\alpha\beta} = 0.$$

4.4 Newtonian Limit of the Geodesic Equation

In the Newtonian theory of gravity the equation of motion for a particle in free fall outside a spherically symmetric body with total mass M is

$$\mathbf{a} = -\nabla\Phi = -\frac{GM}{r^2}\hat{\mathbf{r}}, \tag{4.2}$$

where Φ is the potential and r is the distance from the center of the body. Newton's theory does not hold in general relativity, but it should be recovered as a good approximation when the gravitational field is weak and static while the particles are moving much slower than the speed of light. Recall the geodesic equation, which determines the path taken by a particle in free fall:

$$\frac{d^2 x^\alpha}{d^2\tau} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0.$$

For slow moving particles we have that

$$\frac{dx^0}{d\tau} = c\frac{dt}{d\tau} \gg \frac{dx^i}{d\tau} = \frac{dx^i}{dt}\frac{dt}{d\tau}.$$

So the geodesic equation can be approximated with

$$\frac{d^2x^\alpha}{d^2\tau} + \Gamma^\alpha_{00}\left(c\frac{dt}{d\tau}\right)^2 = 0.$$

For a static field we have that

$$\Gamma^\alpha_{00} = \frac{1}{2}g^{\alpha\mu}(g_{0\mu,0} + g_{\mu 0,0} - g_{00,\mu}) = -\frac{1}{2}g^{\alpha\mu}g_{00,\mu},$$

since $g_{\alpha\beta,0} = 0$. Thus we have that

$$\frac{d^2x^\alpha}{d^2\tau} - \frac{1}{2}g^{\alpha\mu}g_{00,\mu}\left(c\frac{dt}{d\tau}\right)^2 = 0. \tag{4.3}$$

The weak field approximation allows us to write the metric tensor as the Minkowski metric plus a small perturbation term:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}.$$

We will ignore all terms higher than first order. This means that all products of h (or its derivatives) with h (or its derivatives) are neglected. With the condition $\delta^\alpha_\gamma = g^{\alpha\beta}g_{\beta\gamma}$, we must have that

$$g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta},$$

where $h^{\alpha\beta} = \eta^{\alpha\mu}\eta^{\beta\nu}h_{\mu\nu}$. To first order, and from the constancy of the Minkowski metric we see that

$$g^{\alpha\mu}g_{00,\mu} = (\eta^{\alpha\mu} - h^{\alpha\mu})(\eta_{00,\mu} + h_{00,\mu}) = \eta^{\alpha\mu}h_{00,\mu}$$

So equation (4.3) now becomes

$$\frac{d^2x^\alpha}{d\tau^2} = \frac{1}{2}\eta^{\alpha\mu}h_{00,\mu}\left(c\frac{dt}{d\tau}\right)^2.$$

Because $\eta^{\alpha\mu}$ is diagonal, we have that the case $\alpha = 0$ implies that $dt/d\tau$ is constant. We can therefore use

$$\frac{d^2x^\alpha}{d\tau^2} = \frac{d^2x^\alpha}{dt^2}\left(\frac{dt}{d\tau}\right)^2.$$

Only considering the spacelike components, the Minkowski tensor $\eta^{\alpha\mu}$ is equal to $-\delta^{\alpha\mu}$ and we have

$$\frac{d^2x^i}{dt^2} = -\frac{1}{2}c^2\delta^{ij}h_{00,j},$$

where i and j can be 1,2 or 3. But the left hand side of this equation is just the usual components of the 3-acceleration. $h_{00,j}$ is just the gradient of h_{00}, so comparing with equation (4.2) we find that

$$h_{00} = \frac{2\Phi}{c^2},$$

and since $g_{00} = \eta_{00} + h_{00}$ we conclude that

$$g_{00} = 1 + \frac{2\Phi}{c^2} = 1 - \frac{2GM}{c^2 r}.$$

Thus the 00-component of the metric tensor seems to be closely related to the Newtonian potential energy.

4.5 Einstein's Field Equations

In 1915 Einstein formulated the equations that govern how spacetime is being curved by matter and energy. These will be refereed to as *Einstein's field equations* and may be written as

$$G_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}, \tag{4.4}$$

where G is Newton's gravitational constant, $T_{\alpha\beta}$ is the stress energy tensor and $G_{\alpha\beta}$ is the Einstein tensor defined by

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} R\, g_{\alpha\beta}. \tag{4.5}$$

In total there are $4 \times 4 = 16$ equations. But since $G_{\alpha\beta}$ (and $T_{\alpha\beta}$) is symmetric this reduces to 10 independent equations. As we found earlier in section 3.2, the Einstein tensor satisfy $G^{\alpha\beta}{}_{;\alpha} = 0$, which reduces the number even further to $10 - 4 = 6$ independent equations.

Another useful form of Einstein's field equations can be obtained if we take the trace on both sides. The trace of $G_{\alpha\beta}$ is

$$g^{\alpha\beta} G_{\alpha\beta} = g^{\alpha\beta} \left(R_{\alpha\beta} - \frac{1}{2} R\, g_{\alpha\beta} \right) = R - 2R = -R$$

since $g^{\alpha\beta} g_{\alpha\beta} = 4$. If we define $T \equiv T^{\alpha}{}_{\alpha} = g^{\alpha\beta} T_{\alpha\beta}$ we get the relation

$$R = -\frac{8\pi G}{c^4} T.$$

This allows us to write Einstein's field equations in the equivalent form:

$$R_{\alpha\beta} = \frac{8\pi G}{c^4} \left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta} \right). \tag{4.6}$$

Einstein's field equations together with the geodesic equations, which govern how particles only influenced by gravity move, build up the core of general relativity.

It should be noted that Einstein's field equations can not be derived from any underlying principle. Nevertheless, one can make reasonable arguments that this is a good candidate. One of these is that it reduces to the Poisson equation

$$\nabla^2\Phi = 4\pi G\rho,$$

where Φ is the gravitational potential and ρ is the mass density when considering the 00-component in a weak-field approximation [7].

5 The Reissner-Nordström Metric

In this chapter we will first derive the Reissner-Nordström metric and then describe some physics in this metric such as time dilation, gravitational redshift, equations of motion for both a charged and non-charged particle and lastly black holes and event horizons.

The Reissner-Nordström metric is a solution to Einstein's field equations that describes the spacetime around a spherically symmetric non-rotating body with mass M and an electric charge Q. Other than spherical symmetry we also have the assumption that the space is empty from matter (there is only an electromagnetic field). When $Q \to 0$ the metric should approach the Schwarzschild metric. Another property the metric should have is that the spacetime is asymptotically flat. In other words, as the distance from the body approaches infinity, the metric must approach the Minkowski metric.

Because of the spherical symmetry the most natural coordinate system to use is of course the spherical coordinate system. In flat spacetime, the metric when using spherical coordinates is

$$ds^2 = c^2dt^2 - dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\phi^2.$$

For curved spacetime (but still spherically symmetric) this can be generalized to

$$ds^2 = A(t,r)c^2dt^2 - B(t,r)dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\phi^2.$$

One could make the assumption that A and B is independent of time, but we will keep the time-dependence for now as we will find later that A and B must necessarily be independent of time.

Recall Einstein's field equations:

$$R_{\alpha\beta} = \frac{8\pi G}{c^4}\left(T_{\alpha\beta} - \frac{1}{2}Tg_{\alpha\beta}\right).$$

Since we are working in vacuum (no matter) with an electromagnetic field we will use the electromagnetic stress-energy tensor given by

$$T_{\alpha\beta} = \frac{1}{\mu_0}\left(\frac{1}{4}g_{\alpha\beta}F_{\mu\nu}F^{\mu\nu} - g_{\beta\nu}F_{\alpha\mu}F^{\nu\mu}\right). \tag{5.1}$$

Remember also that the electromagnetic stress-energy tensor was traceless, which means that Einstein's field equations can be written as

$$R_{\alpha\beta} = \frac{8\pi G}{c^4}T_{\alpha\beta}.$$

Finally we also need the source-free Maxwell equations given by

$$F^{\alpha\beta}{}_{;\beta} = 0 \tag{5.2}$$

$$F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta} = 0. \tag{5.3}$$

The calculation of the Ricci tensor is done by first calculating the Christoffel symbols with

$$\Gamma^{\lambda}_{\alpha\beta} = \frac{1}{2}g^{\lambda\mu}(\partial_\alpha g_{\beta\mu} + \partial_\beta g_{\mu\alpha} - \partial_\mu g_{\alpha\beta}),$$

and then computing the Ricci tensor with

$$R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta} = \partial_\mu \Gamma^{\mu}_{\alpha\beta} - \partial_\beta \Gamma^{\mu}_{\alpha\mu} + \Gamma^{\mu}_{\nu\mu}\Gamma^{\nu}_{\alpha\beta} - \Gamma^{\mu}_{\nu\beta}\Gamma^{\nu}_{\alpha\mu}.$$

as they were derived in section 3.1 and 3.2, respectively. It is a straightforward but tedious task, so the calculation of both the Christoffel symbols and the Ricci tensor was done with the help of a computer algebra software (Maple 17). All the non-zero Christoffel symbols are given by

$$\begin{aligned}
&\Gamma^0_{00} = \frac{\dot{A}}{2Ac} && \\
&\Gamma^1_{01} = \Gamma^1_{10} = \frac{\dot{B}}{2Bc}, && \Gamma^0_{11} = \frac{\dot{B}}{2Ac} \\
&\Gamma^0_{01} = \Gamma^0_{10} = \frac{A'}{2A}, && \Gamma^1_{00} = \frac{A'}{2B} \\
&\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}, && \Gamma^1_{11} = \frac{B'}{2B} \\
&\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, && \Gamma^1_{22} = -\frac{r}{B} \\
&\Gamma^3_{23} = \Gamma^3_{32} = \cot\theta, && \Gamma^1_{33} = -\frac{r\sin^2\theta}{B} \\
&\Gamma^2_{33} = -\sin\theta\cos\theta &&
\end{aligned}$$

where a dot represent differentiation w.r.t. t, and a prime w.r.t. r. For the Ricci tensor all non-zero components are:

$$R_{00} = -\frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{A''}{2B} + \frac{A'}{Br} - \frac{\ddot{B}}{2Bc^2} + \frac{\dot{B}}{4Bc^2}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right) \tag{5.4}$$

$$R_{11} = \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A''}{2A} + \frac{B'}{Br} - \frac{\ddot{B}}{2Ac^2} - \frac{\dot{B}}{4Ac^2}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right) \tag{5.5}$$

$$R_{22} = -\frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) - \frac{1}{B} + 1 \tag{5.6}$$

$$R_{33} = R_{22}\sin^2\theta \tag{5.7}$$

$$R_{01} = R_{10} = \frac{\dot{B}}{Brc}. \tag{5.8}$$

This is as far as we can get in generalizing a spherically symmetric gravitational field. To determine A and B any further we need to invoke Einstein's field equations, and that means we must specify the stress-energy tensor. In our case the stress-energy tensor was given in terms of the metric tensor and the electromagnetic tensor $F_{\alpha\beta}$. From the spherical symmetry we have that the electric field can only have a radial component. Also, this radial component must not depend on θ or ϕ, so we have that

$$E_r = E_1 = E_1(t,r) = c\,F_{01} = -c\,F_{10}.$$

All the other components are zero since there are no currents or magnetic monopoles. In matrix form we have

$$F_{\alpha\beta} = \begin{bmatrix} 0 & E_r/c & 0 & 0 \\ -E_r/c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The components of the stress-energy tensor can now be computed with equation (5.1). Consider the first term in the parenthesis. Carrying out the summation gives

$$\begin{aligned} \frac{1}{4}g_{\alpha\beta}F_{\mu\nu}F^{\mu\nu} &= \frac{1}{4}g_{\alpha\beta}\left(F_{\mu_0}F^{\mu 0} + F_{\mu_1}F^{\mu 1}\right) = \frac{1}{4}g_{\alpha\beta}\left(F_{10}F^{10} + F_{01}F^{01}\right) \\ &= \frac{1}{4}g_{\alpha\beta}\left(2F_{01}F^{01}\right) = \frac{1}{2}g_{\alpha\beta}F_{01}F^{01}. \end{aligned}$$

For the second term we get

$$g_{\beta\nu}F_{\alpha\mu}F^{\nu\mu} = g_{\beta\nu}F_{\alpha 0}F^{\nu 0} + g_{\beta\nu}F_{\alpha 1}F^{\nu 1} = g_{\beta 1}F_{\alpha 0}F^{10} + g_{\beta 0}F_{\alpha 1}F^{01},$$

and we can write equation (5.1) as

$$T_{\alpha\beta} = \frac{1}{\mu_0}\left(\frac{1}{2}g_{\alpha\beta}F_{01}F^{01} - g_{\beta 1}F_{\alpha 0}F^{10} - g_{\beta 0}F_{\alpha 1}F^{01}\right).$$

The components the stress-energy tensor can now easily be obtained. We have:

$$
\begin{aligned}
T_{00} &= \frac{1}{\mu_0}\left(\frac{1}{2}g_{00}F_{01}F^{01} - g_{00}F_{01}F^{01}\right) \\
&= -\frac{1}{2\mu_0}g_{00}F_{01}F^{01} = -\frac{1}{2\mu_0}A\,F_{01}F^{01} \\
\\
T_{11} &= \frac{1}{\mu_0}\left(\frac{1}{2}g_{11}F_{01}F^{01} - g_{11}F_{01}F^{01}\right) \\
&= -\frac{1}{2\mu_0}g_{11}F_{01}F^{01} = \frac{1}{2\mu_0}B\,F_{01}F^{01} \\
\\
T_{22} &= \frac{1}{2\mu_0}g_{22}F_{01}F^{01} = -\frac{1}{2\mu_0}\,r^2\,F_{01}F^{01} \\
\\
T_{33} &= \frac{1}{2\mu_0}g_{33}F_{01}F^{01} = T_{22}\sin^2\theta.
\end{aligned}
\tag{5.9}
$$

All other (non-diagonal) components of the stress-energy tensor turns out to be zero. Since $T_{01} = 0$ we also have that $R_{01} = 0$ and with equation (5.8) we conclude that $\dot{B} = 0$, which means that B can not depend on t.

Note that

$$\frac{T_{00}}{A} + \frac{T_{11}}{B} = 0.$$

This in turn implies that

$$0 = \frac{R_{00}}{A} + \frac{R_{11}}{B} = \frac{1}{rB}\left(\frac{A'}{A} + \frac{B'}{B}\right).$$

From the above expression we obtain

$$0 = \frac{A'}{A} + \frac{B'}{B} = \frac{\partial}{\partial r}\ln(AB).$$

This means that the product AB must be constant with respect to r. We can write this as

$$AB = f(t),$$

where $f(t)$ is some function that does not depend on r. We will however show later that f must equal unity, but we will keep it as it is for now. As we now have the relation $g_{00} = -f/g_{11}$ (since $A = g_{00}$ and $B = -g_{11}$), we can easily show that

$$F_{01} = g_{00}\,g_{11}F^{01} = -fF^{01}.$$

We will now solve Maxwell's equations. Equation (5.3) does not give us any more information because of the non-existence of magnetic monopoles, and it can be seen that it is directly satisfied by considering the case $\alpha = 0$, $\beta = 1$ and $\gamma = 0$:

$$F_{01;0} + F_{10;0} + F_{00;1} = F_{01;0} - F_{01;0} = 0.$$

Similarly for the other cases it can be showed that $F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta}$ is identically zero. Equation (5.2) will however give us a more explicit form of E_r. Using equation (3.6) for the covariant derivative of a second rank contravariant tensor, equation (5.2) reads

$$0 = F^{\alpha\beta}{}_{;\beta} = \partial_\beta F^{\alpha\beta} + \Gamma^\alpha_{\mu\beta} F^{\mu\beta} + \Gamma^\beta_{\mu\beta} F^{\alpha\mu}. \tag{5.10}$$

For $\alpha = 1$ the above becomes

$$0 = \partial_0 F^{10} + \Gamma^1_{\mu\beta} F^{\mu\beta} + \Gamma^\beta_{\mu\beta} F^{1\mu}.$$

The second term in the above equation vanish

$$\Gamma^1_{\mu\beta} F^{\mu\beta} = \Gamma^1_{\mu 0} F^{\mu 0} + \Gamma^1_{\mu 1} F^{\mu 1} = \Gamma^1_{10} F^{10} + \Gamma^1_{01} F^{01} = 0,$$

since $\Gamma^1_{01} = \Gamma^1_{10} = 0$ (or since $F^{01} = -F^{10}$ together with $\Gamma^1_{01} = \Gamma^1_{10}$). The third term also vanish

$$\Gamma^\beta_{\mu\beta} F^{1\mu} = F^{10}\left(\Gamma^0_{00} + \Gamma^1_{01} + \Gamma^2_{02} + \Gamma^3_{03}\right) = 0$$

since all the Christoffel symbols in the parenthesis equals zero. Thus we simply end up with

$$0 = \partial_0 F^{10}.$$

This of course implies that F^{10}, and therefore E_r, must not depend on time. That is, we have

$$E_r = E_r(r).$$

Using $\alpha = 0$ in equation (5.10) we obtain

$$0 = \partial_1 F^{01} + \Gamma^0_{\mu\beta} F^{\mu\beta} + \Gamma^\beta_{\mu\beta} F^{0\mu}. \tag{5.11}$$

Similarly as before the second term vanish but in this case the third term does not. We have that

$$\begin{aligned}\Gamma^\beta_{\mu\beta} F^{0\mu} &= \Gamma^\beta_{1\beta} F^{01} = F^{01}\left(\Gamma^0_{10} + \Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13}\right) \\ &= F^{01}\left(\frac{A'}{2A} + \frac{B'}{2B} + \frac{2}{r}\right) = \frac{2}{r} F^{01}\end{aligned}$$

since

$$\frac{A'}{2A} + \frac{B'}{2B} = \frac{1}{2}\left(\frac{\partial}{\partial r}\ln(AB)\right) = \frac{1}{2}\left(\frac{\partial}{\partial r}\ln(f)\right) = 0.$$

Equation (5.11) now reads

$$0 = \frac{\partial}{\partial r} F^{01} + \frac{2}{r} F^{01}$$

which is an ordinary first order differential equation with the (easily checked) solution

$$F^{01} = \frac{const.}{r^2},$$

which let us write

$$E_r = \frac{const.}{r^2}.$$

By the Gauss's flux theorem we can conclude that the constant must equal $Q/4\pi\epsilon_0$, and we have

$$E_r = \frac{Q}{4\pi\epsilon_0 r^2}.$$

This is not an unfamiliar expression. It is really Coulomb's law, although one must of course remember that r is just the one of our chosen coordinate and do not necessarily measures the "real" radial distance when in a Reissner-Nordström spacetime.

We are now close to get the final form of the Reissner-Nordström metric. We only need to get a more explicit form of A and B in terms of r. This can now be done by considering one of Einstein's field equations, namely

$$R_{22} = \frac{8\pi G}{c^4} T_{22}.$$

For the left hand side we have

$$R_{22} = -\frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) - \frac{1}{B} + 1 = -\frac{1}{f}\frac{\partial}{\partial r}(rA) + 1$$

which one gets by substituting $B = f/A$ and $B' = -fA'/A^2$ and using the product rule. For the right hand side we use equation (5.9) and obtain

$$-\frac{1}{f}\frac{\partial}{\partial r}(rA) + 1 = \frac{1}{f}\frac{8\pi G}{c^4}\frac{1}{2\mu_0 c^2} r^2 E_r^2.$$

With $E_r^2 = Q^2/(4\pi\epsilon_0 r^2)^2$ this can be written as

$$\frac{\partial}{\partial r}(rA) = f - \frac{GQ^2}{4\pi c^6 \mu_0 \epsilon_0^2 r^2}.$$

If we now integrate and use that $c^2\mu_0 = 1/\epsilon_0$ this becomes

$$A = f + \frac{C(t)}{r} + \frac{GQ^2}{4\pi\epsilon_0 c^4 r^2},$$

where $C(t)$ is a function that may depend on time. When $Q = 0$ the metric must reduce to the Schwarzschild metric. And as we showed in section 4.4, when gravity is weak (i.e. when r is large) the metric tensor component g_{00} must approach $1 - 2GM/c^2r$. So at this limit, if the geodesics of the metric should agree with the motion of Newtonian gravity, we must have that $f = 1$ (which implies that $AB = 1$) and that $C(t) = -2GM/c^2 \equiv -r_s$. It is worth noting that from the relation $AB = f(t)$ one could directly get rid of $f(t)$ by redefining the time coordinate as $dt \to \sqrt{f(t)}dt$.

The constant r_s is commonly known as the *Schwarzschild radius*. If we also define

$$r_Q^2 \equiv \frac{GQ^2}{4\pi\epsilon_0 c^4},$$

A and B can finally be written as

$$\begin{aligned} A &= 1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2} \\ B &= \frac{1}{A} = \left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right)^{-1}, \end{aligned}$$

and the metric tensor in matrix form as

$$g_{\alpha\beta} = \begin{bmatrix} \left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right) & 0 & 0 & 0 \\ 0 & -\left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{bmatrix}.$$

We now have the complete Reissner-Nordström metric derived from Einstein's field equations together with Maxwell's equations. An interesting note however; we did not include any magnetic monopoles in this derivation. There is no experimental evidence of the existence of magnetic monopoles so it is only natural to leave them out. Still, there is no theoretical arguments that they should not exist, so it can be pleasing to assume that they exist and see how the metric would look like if they did.

Let P be the magnetic charge of the body. Besides the radial electric field component we would now also have a radial magnetic field with magnitude B_r. This magnetic field would go like [8]

$$B_r \propto \frac{P}{r^2},$$

which corresponds to the electromagnetic tensor component

$$F_{23} = r^2 B_r \sin\theta.$$

The diagonal components of the stress-energy tensor would also get one extra term containing B_r^2. The implication of this is that we could make our "no magnetic charge"-solution to a "with magnetic charge"-solution by simply replacing Q^2 by $Q^2 + P^2/c^2$, if one uses SI-units where the magnetic charge is measured in ampere·meters ($A \cdot m$).

This discussion about magnetic monopoles was just a quick add-on to the Reissner-Nordström metric but we will not use it any further. That is, in the following sections we will use $P = 0$.

5.1 Gravitational Time Dilation and Redshift

Consider some fixed point in space. With constant r, θ and ϕ we have that dr, $d\theta$ and $d\phi$ is zero and the metric becomes

$$ds^2 = \left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right) c^2 dt^2.$$

Using that $ds^2 = c^2 d\tau^2$ we obtain

$$d\tau = dt \sqrt{1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}}. \tag{5.12}$$

dt can be interpreted as an infinitesimal time interval measured by a observer that is infinitely far away from a gravitational body while $d\tau$ is the interval measured by an observer at a distance r from the center of the body. Suppose that the quantity inside the square root in equation (5.12) is positive and less than unity (this is the case for any physically real situation as discussed briefly in section 5.4). We then have that $d\tau < dt$. This means that the far-away observer will measure the clock that is closer to the body run slower by a factor of $\sqrt{1 - r_s/r + r_Q^2/r^2}$.

Suppose now that an electromagnetic wave is transmitted radially outwards (or inwards) from a point (r_1, θ, ϕ) to another point at (r_2, θ, ϕ). Let t_1 be the coordinate time of emission and t_2 be the coordinate time of reception. The electromagnetic wave is traveling along a radial null geodesic, so we can use $0 = d\theta = d\phi$ and the metric becomes

$$0 = ds^2 = A\, dt^2 - \frac{c^2}{A}\, dr^2.$$

Using λ as an parameter for the geodesic the above can be written as

$$A^2 \left(\frac{dt}{d\lambda}\right)^2 = c^2 \left(\frac{dr}{d\lambda}\right)^2.$$

If we let λ_1 correspond to t_1 and λ_2 to t_2, solving for $dt/d\lambda$ and integrating we obtain

$$t_2 - t_1 = c\int_{\lambda_1}^{\lambda_2} \frac{1}{A}\frac{dr}{d\lambda}\,d\lambda.$$

Let now t_1' be the coordinate time when the electromagnetic wave has oscillated exactly one period after first emission at $r = r_1$. Similarly, let t_2' be the coordinate time when the electromagnetic wave has oscillated one period after receiving the signal at $r = r_2$. Since the integral above does not depend on t we have that

$$t_2 - t_1 = t_2' - t_1',$$

or equivalently as

$$\Delta t_2 \equiv t_2' - t_2 = \Delta t_1 \equiv t_1' - t_1.$$

From this we see that the coordinate time period at the point of emission and the coordinate time period at the point of reception are equal. A real clock however measures the proper time, so with equation (5.12) the periods in proper time are

$$\Delta\tau_1 = \Delta t_1\sqrt{1 - \frac{r_s}{r_1} + \frac{r_Q^2}{r_1^2}}$$

and

$$\Delta\tau_2 = \Delta t_2\sqrt{1 - \frac{r_s}{r_2} + \frac{r_Q^2}{r_2^2}}.$$

The proper frequency of the electromagnetic wave is simply the reciprocal of the proper period and since $\Delta t_1 = \Delta t_2$ we must have the relation

$$f_2 = f_1\left(1 - \frac{r_s}{r_2} + \frac{r_Q^2}{r_2^2}\right)^{1/2}\left(1 - \frac{r_s}{r_1} + \frac{r_Q^2}{r_1^2}\right)^{-1/2}, \tag{5.13}$$

where f_1 and f_2 are the proper frequencies measured by an observer located at r_1 and r_2, respectively.

Suppose that both r_1 and r_2 are large enough so that the quantities inside the square roots are positive. If $r_1 < r_2$ (traveling radially outwards), we have that

$$\left(1 - \frac{r_s}{r_2} + \frac{r_Q^2}{r_2^2}\right)^{1/2} < \left(1 - \frac{r_s}{r_1} + \frac{r_Q^2}{r_1^2}\right)^{1/2}$$

which means that f_2 will be smaller than f_1 and the wave is said to be redshifted. Letting $r_1 > r_2$ (traveling radially inwards) the result would instead be that f_2 is larger than f_1 and the wave is said to be blueshifted.

5.2 The Geodesic Equation

We will now look at the equations that describes the motion of a massive non-charged freely falling particle or a photon in a Reissner-Nordström spacetime. Both the particle and the photon will follow a geodesic, so finding out the path they take is the same as solving the geodesic equation. The massive particle will follow a timelike geodesic while the photon follows a null geodesic. **Note:** in this and the following sections we put $c = 1$. We do this because it let us skip a lot of tedious work remembering to use c when dealing with the t-component.

Let $x^\alpha = x^\alpha(\lambda)$ be a curve parameterized by λ. For x^α to be a geodesic it must satisfy the geodesic equation given by

$$\frac{d^2x^\alpha}{d^2\lambda} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0. \tag{5.14}$$

For the massive particle it would be natural to have the geodesic parameterized by the proper time, and we will actually do this later. But since the photon follows a null geodesic it can not be parameterized by the proper time so we will stick with the more general parameter λ so that we can treat both the massive particle and the photon as far as possible.

Using all the non-zero Christoffel symbols written down in the beginning of this chapter we have that for $\alpha = 0$, equation (5.14) reads

$$\frac{d^2t}{d\lambda^2} + \frac{A'}{A}\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0. \tag{5.15}$$

Considering $\alpha = 1$ we obtain

$$\begin{aligned} 0 = &\frac{d^2r}{d\lambda^2} + \frac{A'}{2B}\left(\frac{dt}{d\lambda}\right)^2 + \frac{B'}{2B}\left(\frac{dr}{d\lambda}\right)^2 \\ &- \frac{r}{B}\left(\frac{d\theta}{d\lambda}\right)^2 - \frac{r\sin^2\theta}{B}\left(\frac{d\phi}{d\lambda}\right)^2. \end{aligned} \tag{5.16}$$

For $\alpha = 2$ and $\alpha = 3$ we get

$$0 = \frac{d^2\theta}{d\lambda^2} + \frac{2}{r}\frac{d\theta}{d\lambda}\frac{dr}{d\lambda} - \sin\theta\cos\theta\left(\frac{d\phi}{d\lambda}\right)^2 \tag{5.17}$$

$$0 = \frac{d^2\phi}{d\lambda^2} + \frac{2}{r}\frac{d\phi}{d\lambda}\frac{dr}{d\lambda} + 2\cot\theta\frac{d\phi}{d\lambda}\frac{d\theta}{d\lambda}. \tag{5.18}$$

By the spherical symmetry we must of course have that the trajectory is in a plane. We can therefore, without any loss of generality, put $\theta = \pi/2$. This means that the derivatives of θ vanish and equation (5.17) is instantly satisfied. With this simplification equation (5.16) and (5.18) now become

$$\frac{d^2r}{d\lambda^2} + \frac{A'}{2B}\left(\frac{dt}{d\lambda}\right)^2 + \frac{B'}{2B}\left(\frac{dr}{d\lambda}\right)^2 - \frac{r}{B}\left(\frac{d\phi}{d\lambda}\right)^2 = 0 \tag{5.19}$$

and

$$\frac{d^2\phi}{d\lambda^2} + \frac{2}{r}\frac{d\phi}{d\lambda}\frac{dr}{d\lambda} = 0, \tag{5.20}$$

respectively. If we divide the above equation (5.20) with $d\phi/d\lambda$ and note that

$$\left(\frac{d\phi}{d\lambda}\right)^{-1}\frac{d^2\phi}{d\lambda^2} = \frac{d}{d\lambda}\ln\left(\frac{d\phi}{d\lambda}\right)$$

and

$$\frac{2}{r}\frac{dr}{d\lambda} = \frac{d}{d\lambda}\ln\left(r^2\right)$$

we obtain

$$\frac{d}{d\lambda}\ln\left(r^2\frac{d\phi}{d\lambda}\right) = 0.$$

This means that the quantity inside the logarithm is a constant of motion and we put it equal to a constant L:

$$r^2\frac{d\phi}{d\lambda} = L. \tag{5.21}$$

If we compare this with the Newtonian theory, L correspond to angular momentum per unit mass. So this equation basically states that angular momentum is conserved.

In a similarly way, we can divide equation (5.15) by $dt/d\lambda$ and obtain

$$\frac{d}{d\lambda}\ln\left(\frac{dt}{d\lambda}A\right) = 0,$$

which again means that the quantity inside the logarithm must be a constant which we will denote by e:

$$\frac{dt}{d\lambda}A \equiv e. \tag{5.22}$$

The interpretation of the constant of motion e will be made later. We can now use equation (5.21) and (5.22) in equation (5.19) and obtain

$$\frac{d^2r}{d\lambda^2} + \frac{e^2A'}{2BA^2} + \frac{B'}{2B}\left(\frac{dr}{d\lambda}\right)^2 - \frac{L^2}{Br^3} = 0.$$

Multiplying the above equation with $2B\,dr/d\lambda$ we can write it as

$$\begin{aligned} 0 &= 2B\,\frac{dr}{d\lambda}\frac{d^2r}{d\lambda^2} + e^2\frac{A'}{A^2}\frac{dr}{d\lambda} + B'\,\frac{dr}{d\lambda}\left(\frac{dr}{d\lambda}\right)^2 - 2\frac{L^2}{r^3}\frac{dr}{d\lambda} \\ &= \frac{d}{d\lambda}\left[B\left(\frac{dr}{d\lambda}\right)^2 - \frac{e^2}{A} + \frac{L^2}{r^2}\right]. \end{aligned}$$

The expression inside the square bracket must be a constant

$$B\left(\frac{dr}{d\lambda}\right)^2 - \frac{e^2}{A} + \frac{L^2}{r^2} = const \equiv -e_0^2,$$

where we implicitly defined e_0 (it will be clear later why we defined it this way). Since $B = 1/A$, an equivalent form to the above is

$$\left(\frac{dr}{d\lambda}\right)^2 = e^2 - A\left(e_0^2 + \frac{L^2}{r^2}\right). \tag{5.23}$$

We now have $dr/d\lambda$ as a function of r alone. But we are also able to express $dr/d\phi$ as a function of r. If we divide the above equation with $\dot{\phi}^2 = L^2/r^4$ we get

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4k^2}{L^2} - r^2A\left(1 + \frac{r^2e_0^2}{L^2}\right). \tag{5.24}$$

Recall that A was defined as

$$A = \left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right),$$

so we have that

$$\begin{aligned} r^2A\left(1 + \frac{r^2e_0^2}{L^2}\right) &= \left(r^2 - rr_s + r_Q^2\right)\left(1 + \frac{r^2e_0^2}{L^2}\right) \\ &= r_Q^2 - r_sr + \left(1 + \frac{r_Q^2e_0^2}{L^2}\right)r^2 - \frac{r_se_0^2}{L^2}r^3 + \frac{e_0^2}{L^2}r^4. \end{aligned}$$

Substituting this into equation (5.24) we finally get

$$\left(\frac{dr}{d\phi}\right)^2 = -r_Q^2 + r_sr - \left(1 + \frac{r_Q^2e_0^2}{L^2}\right)r^2 + \frac{r_se_0^2}{L^2}r^3 - \frac{1}{L^2}\left(e_0^2 - e^2\right)r^4 \tag{5.25}$$

We have now obtained $dr/d\phi$ in terms of r. But as will be shown below this can be simplified if we look at two special cases, namely null geodesics and timelike geodesics corresponding to paths taken by massless and massive particles, respectively.

Remember that when $c = 1$ and $\theta = \pi/2$, the metric is given by

$$ds^2 = d\tau^2 = A\,dt^2 - \frac{1}{A}\,dr^2 - r^2\,d\phi^2.$$

From equation (5.21), (5.22) and (5.23) we have that

$$\begin{aligned} d\phi^2 &= \frac{L^2}{r^4}d\lambda^2 \\ dt^2 &= \frac{e^2}{A^2}d\lambda^2 \\ dr^2 &= \left[e^2 - A\left(e_0^2 + \frac{L^2}{r^2}\right)\right]d\lambda^2 \end{aligned}$$

which let us write the differential proper time simply as

$$d\tau^2 = e_0^2 \, d\lambda^2.$$

From this we see that for null geodesics e_0 must equal zero, and for timelike geodesics e_0 must be larger than zero. So for timelike geodesics parameterized with the proper time, e_0 simply equals unity and equation (5.25) reads

$$\left(\frac{dr}{d\phi}\right)^2 = -r_Q^2 + r_s r - \left(1 + \frac{r_Q^2}{L^2}\right) r^2 + \frac{r_s}{L^2} r^3 - \frac{1}{L^2}\left(1 - e^2\right) r^4, \tag{5.26}$$

while for null geodesics equation (5.25) reduces to

$$\left(\frac{dr}{d\phi}\right)^2 = -r_Q^2 + r_s r - r^2 + \frac{e^2}{L^2} r^4. \tag{5.27}$$

5.2.1 Comparison to Newtonian Mechanics

We are now going to make a comparison of the recently found equation of motion with the well known case of Newtonian mechanics. Using $Q = 0$, we can rewrite equation (5.23) as

$$e^2 - e_0^2 = \left(\frac{dr}{d\tau}\right)^2 + \frac{L^2}{r^2} - e_0^2 \frac{2GM}{r} - \frac{2GML^2}{r^3}.$$

This can be considered as an energy equation where the first two terms on the R.H.S. correspond to the kinetic part and the third term to the potential part. In Newtonian mechanics the total energy E of a particle with mass m that is influenced only by gravity from a spherically symmetric object with mass M satisfies

$$\frac{2E}{m} = \dot{r}^2 + r^2\dot{\phi}^2 - \frac{2GM}{r} = \dot{r}^2 + \frac{L^2}{r^2} - \frac{2GM}{r}.$$

By comparing the two above equations we can see that $e^2 - e_0^2$ corresponds to the total energy per unit mass. But since E is the Newtonian total energy (i.e. no rest energy) we can make the interpretation that e^2 corresponds to the total relativistic energy per unit mass while e_0^2 corresponds to the rest energy per unit mass of the particle. So our conclusion earlier that $e_0 = 1$ (when using τ as the parameter) for massive particles and $e_0 = 0$ for massless particles now have an intuitive understanding. The big difference from Newtonian mechanics is that general relativity introduces an extra term which depend on r^{-3} which get dominant when r is small. When r is large (i.e. for weak gravity) this extra term can be neglected and the equation reduces to the Newtonian one.

5.2.2 Circular Orbits of Photons

One interesting special case is that photons (or other massless particles) can be in a circular orbit. For circular orbits we must have $\frac{dr}{d\lambda} = 0$ and $\frac{d^2r}{d\lambda^2} = 0$. From equation (5.19) we must have that

$$\frac{A'}{2}\left(\frac{dt}{d\lambda}\right)^2 - r\left(\frac{d\phi}{d\lambda}\right)^2 = 0$$

while from the metric (since $d\tau = 0$) we get

$$A\left(\frac{dt}{d\lambda}\right)^2 - r^2\left(\frac{d\phi}{d\lambda}\right)^2 = 0.$$

From the two expressions above we must therefore have that $rA' = 2A$, which explicitly reads

$$\frac{r_s}{r} - \frac{2r_Q^2}{r^2} = 2\left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right).$$

This can be written as the quadratic equation

$$0 = r^2 - \frac{3}{2}r_s r + 2r_Q^2$$

with the solution

$$r_\pm = \frac{3r_s}{4} \pm \sqrt{\left(\frac{3r_s}{4}\right)^2 - 2r_Q^2}.$$

We have three different cases depending on the value of the expression inside the square root. These three cases correspond to the existence of two, one or zero real-valued solutions. The only possible orbit for the photon is however the solution given by r_+, as will be clear when discussing black holes and event horizons later in section 5.4. The reason is that r_- will be inside of the event horizon except in the case when $2r_Q = r_s$, then the r_- photon orbit and the event horizon are at the same radius.

5.3 Motion of a Charged Particle

In the Reissner-Nordström spacetime we have a static electric field, so asking how the motion of a charged particle would be is natural. We will now derive the equations describing this charged particle with help of the lagrangian formalism, which we assume that the reader is familiar to.

The Lagrangian for a charged particle is [10]

$$\mathcal{L} = \frac{1}{2}g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta + qA_\alpha\dot{x}^\alpha$$

where q is the charge per unit mass of the particle, A_α the four-potential and a dot represents a differentiation with respect to the proper time. If we had no electromagnetic field present the Lagrangian would only consist of the first term. In flat spacetime this first term would be the kinetic part (using $\eta_{\alpha\beta}$ instead of $g_{\alpha\beta}$), while in a gravitational field it also contains the gravitational potential energy. The second term arises because we have potential energy from the electromagnetic field, and splitting it up in its timelike and spacelike coordinates we have

$$qA_\alpha \dot{x}^\alpha = qA_0 u^0 + qA_i u^i = q\Phi u^0 + qA_i u^i,$$

where u^α is the 4-velocity. The first term in this expression clearly corresponds to the static electric potential energy while the second term corresponds to a "magnetic potential energy". If one make a gauge transformation of the electric and magnetic potentials the Lagrangian will change by an addition of a total derivative of a function. This will however not change the equations of motion.

Since we only have a radial electric field and no magnetic field the only non-zero component of the four-potential is

$$A_0 = \frac{Q}{4\pi\epsilon_0 r} \equiv \frac{\tilde{Q}}{r}$$

and the Lagrangian now becomes

$$\mathcal{L} = \frac{1}{2}\left[A\,\dot{t}^2 - A^{-1}\,\dot{r}^2 - r^2\,\dot{\theta}^2 - r^2\sin^2\theta\,\dot{\phi}^2\right] + \frac{q\,\tilde{Q}}{r}\,\dot{t}.$$

The motion of the particle is determined by solving the Euler-Lagrange equations given by

$$\frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\alpha}\right) - \frac{\partial\mathcal{L}}{\partial x^\alpha} = 0.$$

Since the Lagrangian does not explicitly depend on t or ϕ we see from the above equation that $\partial\mathcal{L}/\partial\dot{t}$ and $\partial\mathcal{L}/\partial\dot{\phi}$ are both constants:

$$\frac{\partial\mathcal{L}}{\partial\dot{t}} = A\,\dot{t} + \frac{q\,\tilde{Q}}{r} \equiv e = const \tag{5.28}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = r^2\,\dot{\phi} \equiv L = const. \tag{5.29}$$

The only difference between equation (5.28) and the equation for e in the case of a massive non-charged particle is the "extra" term $q\tilde{Q}/r$ which clearly corresponds to a static electric potential energy. As before, the particle must move in a plane and we can put $\theta = \pi/2$. So for the θ-component ($\alpha = 2$) the

Euler-Lagrange equation is immediately satisfied and for the r-component ($\alpha = 1$) we get

$$
\begin{aligned}
0 &= \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) - \frac{\partial \mathcal{L}}{\partial r} \\
&= -A^{-1}\,\ddot{r} - \frac{1}{2}\left(A'\,\dot{t}^2 + A^{-2}A'\,\dot{r}^2 - 2r\dot{\phi}^2\right) - \frac{q\,\tilde{Q}}{r^2}\dot{t}
\end{aligned}
$$

which is the same as equation (5.19) except for the last term containing $\tilde{Q}$. We will however not use the above equation. Instead we will use the metric which, when $\theta = \pi/2$, is given by

$$
ds^2 = d\tau^2 = A\,dt^2 - A^{-1}\,dr^2 - r^2 d\phi^2.
$$

If we divide by $d\tau^2$ and multiply with A this can be written as

$$
\dot{r}^2 + A - A^2\,\dot{t}^2 + A\,r^2\dot{\phi}^2 = 0.
$$

Using equation (5.28) and (5.29) we obtain

$$
\begin{aligned}
0 &= \dot{r}^2 + A - \left(e - \frac{q\,\tilde{Q}}{r}\right)^2 + A\,\frac{L^2}{r^2} \\
&= \dot{r}^2 + A\left(1 + \frac{L^2}{r^2}\right) - \left(e - \frac{q\,\tilde{Q}}{r}\right)^2.
\end{aligned}
$$

By dividing the above equation with $\dot{\phi}^2 = L^2/r^4$ we can write it as

$$
\begin{aligned}
\left(\frac{dr}{d\phi}\right)^2 &= -r^2 A\left(1 + \frac{r^2}{L^2}\right) + \frac{r^4}{L^2}\left(e - \frac{q\,\tilde{Q}}{r}\right)^2 \\
&= -r_Q^2 + r_s\,r - \left(1 + \frac{r_Q^2 - q^2\tilde{Q}^2}{L^2}\right) r^2 \\
&\quad + \frac{1}{L^2}\left(r_s - 2eq\tilde{Q}\right) r^3 - \frac{1}{L^2}\left(1 - e^2\right) r^4.
\end{aligned}
$$

This is the equation that govern the motion of a charged particle in the Reissner-Nordström metric. And as it must be, setting $q = 0$ reduces this to equation (5.26) which describes a free falling non-charged massive particle.

5.4 Event Horizons and Black Holes

To build up the discussion in a logical way, we will in this section first consider the Schwarzschild black hole and then proceed to the Reissner-Nordström black hole. The Schwarzschild metric (when $c = 1$) is given by

$$
ds^2 = g_{tt}dt^2 + g_{rr}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta\,d\phi^2
$$

where

$$g_{tt} = -\frac{1}{g_{rr}} = 1 - \frac{r_s}{r}.$$

The metric becomes singular at $r = r_s$ because when $r \to r_s$ we have that $g_{rr} \to \pm\infty$. Another singular point is clearly $r = 0$. Note that for any spherically symmetric object (e.g. a star) with a radius larger than r_s there will be no singularity. This is because the Schwarzschild solution is only valid outside the object where there is vacuum, which imply that all components of the stress-energy tensor are zero. Inside the object the stress-energy tensor does not vanish and one would obtain a solution that does not have any singularities.

Consider now an object that do have a radius less than r_s. Then the Schwarzschild solution does hold at $r = r_s$ and we do have a singularity at $r = r_s$. An object with the property that its radius is less than r_s is called a *black hole*. This name is justified (as will be shown later) by the fact that no massive particle nor light can escape if at a distance closer than r_s. This "boundary" in spacetime is called an *event horizon* and it marks the surface for which events inside of it can not affect the outside.

As discussed in section 5.1, gravitational time dilation and redshift will appear in a gravitational field. When dealing with black holes a few interesting things happen. To an observer, a clock near the black hole will appear to run slower than a clock further away. By equation (5.12) (letting $r_Q = 0$), one can see that the time dilation will be infinite for a clock falling towards the black hole as it approaches the event horizon. This leads to that it would require an infinite time to reach the event horizon for an object falling towards the black hole, as seen from an outside observer. Also, from equation (5.13) one can see that the redshift of an electromagnetic wave traveling outwards goes to infinity when the point of emission approaches the event horizon.

Up till now we have worked with the spherical coordinates (t, r, θ, ϕ). But what if we were to choose a different coordinate system, would there still be any singularities? As it turns out, the singularity at $r = r_s$ can actually be made to disappear if one uses the right set of coordinates, and one can see that it would not take an infinite amount of proper time for an object to fall past the event horizon. To see that $r = r_s$ is merely an apparent singularity and that not even light can escape from a black hole, we will use the so-called Lemaître coordinates λ and ρ which differentials transform as

$$d\lambda = dt + \sqrt{\frac{r_s}{r}}\left(1 - \frac{r_s}{r}\right)^{-1} dr$$

$$d\rho = dt + \sqrt{\frac{r}{r_s}}\left(1 - \frac{r_s}{r}\right)^{-1} dr.$$

Taking the difference between the two above equations we have

$$d\rho - d\lambda = \left(\sqrt{\frac{r}{r_s}} - \sqrt{\frac{r_s}{r}}\right)\left(1 - \frac{r_s}{r}\right)^{-1} dr = \sqrt{\frac{r}{r_s}} dr. \tag{5.30}$$

Integrating the above results in

$$r = r_s^{1/3}\left(\frac{3}{2}(\rho - \lambda)\right)^{2/3},$$

and one will find that the metric can be written as

$$ds^2 = d\lambda^2 - \frac{r_s}{r} d\rho^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2. \tag{5.31}$$

With this metric we see only one singularity located at $r = 0$, while no singularity arise at $r = r_s$. However this singularity at the center will not disappear by any transformation, it is really a true singularity.

Consider now a photon that travels along a radial trajectory. We have that $ds^2 = d\theta^2 = d\phi^2 = 0$ and equation (5.31) can be written as

$$d\lambda = \pm\sqrt{\frac{r_s}{r}} d\rho$$

where a plus sign correspond to the photon traveling outward while a negative sign correspond to inward motion. Using the above and equation (5.30) one can obtain

$$dr = \left(\pm 1 - \sqrt{\frac{r_s}{r}}\right) d\lambda.$$

From this we see that if $r < r_s$, the expression inside the parenthesis is negative which means that dr is always negative (if $d\lambda$ is positive). So photons, regardless if they were emitted outward or inward, end up at the center of the black hole. We did assume that the photon was traveling radially, so strictly the conclusion of course only hold for that special case. But if a photon that was emitted directly outward just inside the event horizon can not escape, it should not be to hard to believe that a photon emitted in any direction (when $r < r_s$) would also be forced to travel towards the center.

We will now consider a Reissner-Nordström black hole. For the Schwarzschild black hole we had one event horizon which, when using the coordinates (t, r, θ, ϕ), we localized by finding singularities in the metric. Recall the Reissner-Nordström metric given by

$$ds^2 = g_{tt} dt^2 + g_{rr} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2$$

where

$$g_{tt} = -\frac{1}{g_{rr}} = 1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}.$$

A quick inspection suggest that possible event horizons should occur when $0 = g_{tt} = 1 - r_s/r + r_Q^2/r^2$ which yields the quadratic equation

$$0 = r^2 - r_s\, r + r_Q^2$$

with the solutions

$$r_{\pm} = \frac{1}{2}\left(r_s \pm \sqrt{r_s^2 - 4\, r_Q^2}\right).$$

From this we see that, depending on the relative values of r_s and r_Q, there is two, one or zero real-valued solutions. A schematic plot of g_{tt} for the three different cases is showed in figure 1.

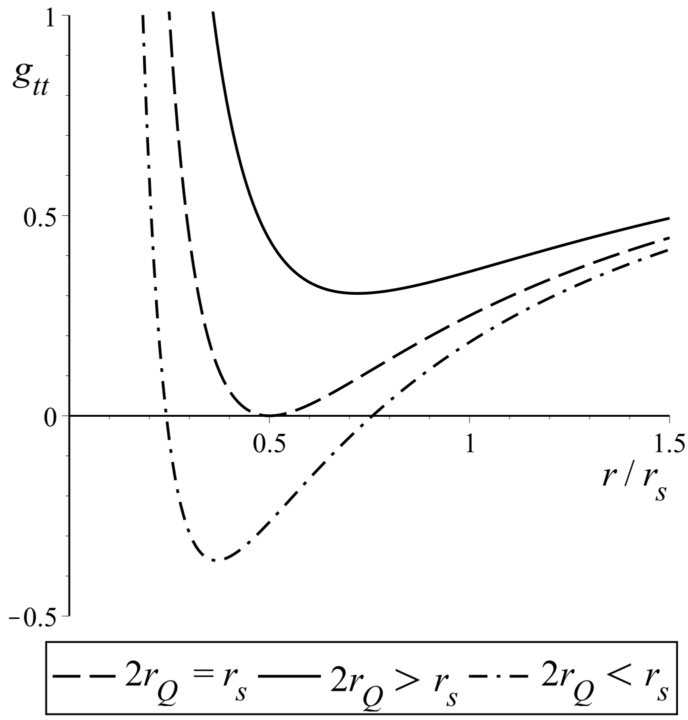

Figure 1: The function $g_{tt}(r) = 1 - r_s/r + r_Q^2/r^2$ for the three different cases. $g_{tt} = 0$ indicates that there is an event horizon.

Consider the situation when $r_s > 2\, r_Q$. In this case there is two coordinate singularities at r_+ and r_-. Note that these two singularities occur because of our choice of coordinate system. As in the Schwarzschild metric one can choose a coordinate system in which there exist no singularities except at $r = 0$ (which still is a true singularity). The metric in this case can be divided into three regions:

$$\begin{aligned} &\text{Region 1:} \quad r_+ < r < \infty \\ &\text{Region 2:} \quad r_- < r < r_+ \\ &\text{Region 3:} \quad 0 < r < r_- \end{aligned}$$

An object coming from region 1 and falling into region 2 would have the same experience as when crossing the event horizon in the Schwarzschild black hole. For an outside observer the infalling object would be infinitely redshifted and it would never reach the event horizon. The proper time for reaching the event horizon for the object would however be finite. Once inside region 2 all massive particles and photons necessarily move in the direction of decreasing r. This unavoidable decrease in r does however stop when reaching region 3, and the object is therefore not doomed to end up at the singularity at $r = 0$. If now moving back (increasing r) to region 2 again, the object can only move in direction of increasing r and ultimately come out beyond the horizon at $r = r_+$ [8]. However, it is highly speculative if this journey through the black hole is physically real.

Consider now the situation when $r_s < 2\,r_Q$. In this case there is no singularities when $r > 0$, and therefore no event horizons. The singularity at $r = 0$ does still exist, which means that there is no event horizon preventing someone far away to directly observe this singularity. A singularity with this property (i.e. no event horizon "hiding" it) is called a *naked singularity*. It is widely believed, but not proven, that no naked singularity (except maybe the one occurring in the Big Bang model) exist in the universe [11]. This assumption is called the *weak cosmic censorship hypothesis* and was formulated by Roger Penrose in 1969. So this solution when $r_s < 2\,r_Q$ is therefore usually considered to be unphysical.

Lastly, when $r_s = 2\,r_Q$ there exist only one horizon, and the black hole is called extremal. The event horizon is located at $r = r_s/2$, and in this case g_{tt} is positive on both sides which means that an observer inside the event horizon does not necessarily move towards the singularity at $r = 0$. However, this extremal black hole with only one horizon seems to be unstable since adding any nonzero mass would turn it to the "regular" case when $r_s > 2\,r_Q$.

6 Summary and Conclusion

In this paper we solved the Einstein-Maxwell equations for a spherically symmetric charged body and found the Reissner-Nordström metric given by

$$ds^2 = \left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right) c^2 dt^2 - \left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right)^{-1} dr^2 - r^2 d\Omega^2.$$

It was later shown how this gravitational field give rise to phenomena such as gravitational time dilation and redshift. With the use of the geodesic equations we derived the equations of motion for both massive and massless non-charged particles while we used the lagrangian formalism to derive the equations of motion for massive charged particles. We also found out how these equations corresponds to the well known equations derived from Newtonian mechanics. Lastly we discussed the properties of a Reissner-Nordström black hole. Depending on the relative value of r_s and r_Q there is two, one or zero event horizons that corresponds to apparent singularities in the metric. The only true singularity is found in the center of the black hole.

The Reissner-Nordström metric is a generalization to the Schwarzschild metric, it can however itself be generalized to the so-called Kerr-Newman metric. It is a solution to the Einstein-Maxwell equations for an electrically charged rotating axially symmetric body. That is, in addition to the Reissner-Nordström metric it has a non-zero angular momentum which has the consequence that it no longer exhibit spherical symmetry. Some methods that we used in deriving the Reissner-Nordström metric can also be used when deriving the Kerr-Newman metric.

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